Optimal Growth and Dynamics with Non-Separable Preferences*

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Abstract

From psychology we know that repetition generates addiction to habits and that envy is an important motive of human behaviour. Economists have borrowed these two principles to build a more realistic model of consumption, in which both addiction and envy are both formalised with the assumption of non-separable preferences. Despite the large amount of literature using the assumption of non-separable preferences, little work has been done to investigate the effect of non-separability on dynamics and optimal growth. Moreover, no work has been done to analyse the different effects on optimal growth and dynamics due to *joint* impact of intragenerational and intergenerational non-separability.

The goal of this paper is to check how optimal growth and dynamics change in an overlapping-generations model *a là* Diamond (1956), modified by intra- and intergenerational non-separable preferences. We show that convergence to the steady state is no longer ensured and that the optimal solution may display locally explosive dynamics under specific conditions on the trace of the Jacobian matrix. We also show that the optimal solution may *not* display damped oscillations, even when the social planner does not discount the utility of future generations, i.e. in the golden rule case.

We prove that these results crucially depend on the *joint* assumption of intragenerational and intergenerational non-separability. In particular, we show that the assumption of separability across periods of life is a necessary and sufficient condition for locally converging oscillatory dynamics.

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Repetition of a stimulus diminishes the perception of the stimulus and the response to it.

Campbell and Cochrane (1999), p.208

When we exclude income, prices and autonomous changes in tastes as possible causes of growth of sales of new product, the only possible cause is a shift in preferences, generated by interdependence in preferences.

Duesenberry (1959)

1 Introduction

From psychology we learn that repetition generates addiction to habits and that envy is an important motive of human behaviour. Economists have borrowed these two principles to build a more realistic model of consumption. Both addiction and envy are formalised with the assumption of non-separable preferences. In the event of addiction, preferences are *intra*generational non-separable: individual satisfaction at time t depends on consumption at time t and on individual consumption at time $t-1$. In the event of envy, preferences are *inter*generational non-separable: individual satisfaction at time t depends on current consumption at time t and aggregate consumption at time $t-1$.

The assumption of non-separable preferences has been used in a wide variety of economic applications. The assumption of non-separable preferences across generations is mostly used to study the effects of peer-groups and social distance.[1] On the other hand, the assumption of non-separable preferences across generation and that of non-separable preferences across periods of life are indifferently used to solve equity premium and risk-free rate puzzles,[2] to explain excess smoothness and excess sensitivity puzzles,[3] to study growth and relationship with saving,[4] to analyse the effects of tax policy,[5] to investigate

[1] Among others, see Akerlof (1997) and Brock and Durlauf (2001).

[2] Campbell and Cochrane (1999) and Abel (1990) use the assumption of non-separable preferences across periods of life, while Constantinides (1990) and Boldrin *et al.* (2001) assume that preferences are non-separable across generations.

[3] Fuhrer (2000) extends a monetary-policy model to incorporate the assumption of addiction to habits and to explain the "humped-shaped" response of consumption to income. Amato and Laubach (2004) use Fuhrer (2000) model and study implications for the optimal policy and the characteristics of the policy change depending on the degree of habits. Michaelides (2002) simulate excess smoothness and excess sensitivity in a buffer-stock saving model with heterogeneous agents and non-separable preferences.

[4] Lahiri and Puhakka (1998) and Wendner (2002) study the effects of addiction on saving. In general, addiction has a negative effect on saving, both in an exchange and ion a productive economy. Carroll *et al.* (2000) investigate the relashionship between saving and growth and they find that habits have a positive effect on growth.while de la Croix and Michel (1999) study the effects on saving and growth in presence of preferences intergenerationally non-separable. They find that the effect on saving is negative, while the effect on growth depends on critical values of some parameters.

[5] Ljungqvist and Uhlig (2000) show that, with non-separable preferences, optimal tax policy affects economy counter-cyclically via pro-cyclical taxes; Lahiri and Puhakka (1998) prove that

steady state implications on wealth distribution,[6] to measure the impact of non-separable preferences on portfolio decisions.[7]

Despite this large literature on non-separable preferences, little work has been done to investigate the effect of non-separability on dynamics and optimal growth.[8] Moreover, no work has been done to analyse the different effects on optimal growth and dynamics due to *joint* impact of intragenerational and intergenerational non-separability.

The justification for including these joint assumptions is obvious when we consider consumption of non-durable goods such as cigarettes or alcohol. In fact, consumption grows not only because of copycatting a behaviour (across-generations effect) but also because of addiction to the consumption itself (across-periods effect).

Given that, the goal of this paper is to check how optimal growth and dynamics change in an overlapping-generations model *a là* Diamond (1956), modified by intra- and intergenerational non-separable preferences.

We show that convergence to the steady state is no longer ensured and that the optimal solution may display locally explosive dynamics under specific conditions on the trace of the Jacobian matrix. We also show that the optimal solution may *not* display damped oscillations, even when the social planner does not discount the utility of future generations, i.e. in the golden rule case.

We prove that these results crucially depend on the *joint* assumption of intragenerational and intergenerational non-separability. In particular, we prove that the assumption of separability across periods of life is a necessary and sufficient condition for locally converging oscillatory dynamics. The intuition for this result is very simple: optimal solution is generally characterised by monotonic convergence if the utility is separable across generations and across periods of life. If the utility is non-separable across generations, but still separable across periods of life, converging oscillatory dynamics are possible; if the utility is non-separable across periods of life, cycles occur; finally, if the utility is non-separable across generations and across periods of life, oscillatory dynamics and cycles give rise to explosive dynamics.[9]

The rest of the paper is organised as follows. Next section presents the main assumptions of the model. Sections 3 and 4 analyse competitive and optimal

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the presence of habits increases saving implying that government is able to float higher level of deficits; Carroll (2000) shows that, in presence of habits, a permanent cut in taxes stimulates aggregate demand less than in a permanent income hypothesis model. A proof is given by the Japanese experience.

[6] Diaz *et al.* (2003) want to understand the role of non-separable preferences in determining precautionary saving volume and wealth distribution shape.

[7] While Galí (1994) consider a general equilibrium model, Abel (1990) an asset pricing model.

[8] With the exception of Lahiri and Puhakka (1998), Michel and Venditti (1997) and Wendner (2002) under the assumption of intragenerational addiction and of Ryder and Heal (1973), del aCroix (1996) and de la Croix and Michel (1999) under the assumption of intergenerational addiction.

[9] Note that these results are defined as *possible* and not as *unique* as they depend on the assumptions on the trace. Changing the assumptions on the trace, in fact, the optimal solution becomes stable in the saddle point sense.

solutions, while section 5 yields a numerical example to illustrate the main results of the paper. Section 6 concludes.

2 The model

The economy is populated by two generations, each one living two periods. We assume that the rate of growth of population is zero and that each young generation is endowed with L_t units of labour.

Under the assumption of non-separable preferences, *(i)* individual sense of well-being is more related to changes in consumption than absolute level of consumption and *(ii)* agents gain utility from absolute consumption and from consumption of the other generation. Thus, the intertemporal utility function is

$$U\left(c_t^1, c_{t+1}^2; e_t, h_{t+1}\right) = u\left(c_t^1, e_t\right) + v\left(c_{t+1}^2, h_{t+1}\right)$$

where c^1 is consumption of the young, c^2 is consumption of the old, e identifies the envy to others, and h the addiction to habits. Envy is formalised as the stock of past consumption of the "active" generation, i.e. $e_t = c_{t-1}^1$. Similarly, habits are formalised as the stock of individual own past consumption, i.e. $h_{t+1} = c_t^1$. For the young, we set $h_t = 0$ by definition without any loss of generality. In other words, we assume that intergenerational and intragenerational interactions work separately on the instantaneous utility functions $u(\cdot)$ and $v(\cdot)$. Moreover, we assume that the stock e is an externality.

We assume that $u_{c^1}, v_{c^2} > 0$, $u_e, v_h < 0$, $u_{c^1c^1}, u_{ee}, v_{c^2c^2}, v_{hh} < 0$ and $u_{c^1e}, v_{c^2h} > 0$. The second and fourth sets of inequalities define reaction to the stocks e and h. Following de la Croix and Michel (1999), we assume that across-generations non-separability has a negative impact on the individual felicity function, i.e. $u_e < 0$. Moreover, as the gap between desired and actual consumption is negative, we assume that individuals catch up with other groups in the society, so that $u_{c^1e} > 0$. Similar comments hold for the assumptions $v_h < 0$ and $v_{c^2h} > 0$.[10]

The first and third sets of assumptions ensure strictly concave utility function, i.e. non-negative marginal utility. As Klijn (1977) and Chapman (1998) point out, once we assume non-separable preferences, it is necessary to restrict the set of admissible solutions in order to have a non-negative marginal utility. A sufficient condition to ensure non-negative marginal utility is that "effective" consumption is always non-negative: $\left(c_t^1 - \theta e_t\right) > 0$ and $\left(c_{t+1}^2 - \delta h_{t+1}\right) > 0$, where $\theta \in [0, 1)$ is the parameter measuring intensity of across-generations non-separability and $\delta \in [0, 1)$ intensity of across-periods non-separability. For the effects on saving and steady state capital intensity, it is important to note that a consequence of this statement is that consumption «will generate a higher level of precautionary saving and a potentially greater level of aggregate consumption smoothness» (Michaelides (2002), p.38, footnote 20).

[10] In this scenario, we generally deal with non-durables (e.g. clothing, food, accessorisies) and the u_{c^1e} measures individual reaction to the good (envy), while v_{c^2h} measures individual reaction to past consumption (addiction).

The above constraint on consumption variables can be re-interpreted in terms of restrictions on δ. In particular, the possibility of a negative marginal utility is ruled out when the indifference curve is downward sloping. The slope of the indifference curve is[11]

$$\frac{\mathrm{d}c_{t+1}^2}{\mathrm{d}c_t^1} = -\frac{u_{c^1} + v_h \frac{\partial h_{t+1}}{\partial c_t^1}}{v_{c^2}} = -\frac{u_{c^1} - \delta v_{c^2}}{v_{c^2}}$$

and the indifference curve is downward sloping if $(u_{c^1} - \delta v_{c^2})/v_{c^2} > 0 \iff u_{c^1}/v_{c^2} > \delta$. If $\delta = 0$, we get the standard model with separable preferences and downward slope is ensured by the set of assumptions on marginal utility: $u_{c^1}, v_{c^2} > 0$. If $\delta > 0$, the slope is negative only if δ is smaller than the critical threshold $\eta \equiv u_{c^1}/v_{c^2}$. As $\delta \geq \eta$, the slope becomes flat when $u_{c^1} = \delta v_{c^2}$ $(\delta = \eta)$ and upward sloping when $u_{c^1} < \delta v_{c^2}$ $(\delta > \eta)$. Therefore, we can restrict the range of δ to $[0, \eta)$ while no additional restrictions are necessary for the domain of θ.

The production function is defined as $Y_t = F(K_t, L_t)$, where K and L are capital and labour employed in total production Y. In per-capita terms, it becomes $y_t = f(k_t)$, in which $f_k > 0$, $f_{kk} < 0$ and Inada conditions are met.

The resource constraint $y_t = c_t^1 + c_t^2 + k_{t+1}$ closes the model setup: total production can be either consumed or invested, under the assumption that capital completely depreciates over time and that the economy is endowed with k_0 and $e_0 = c_{-1}$ at time $t = 0$.

3 Competitive equilibrium

Inputs are paid their marginal products and firm earns zero profits: $R_t = f_k(k_t)$ and $w_t = f(k_t) - k_t f_k(k_t)$, where R is the interest factor on capital and w is the real wage. Some observations: *(i)* firm's programme is static; *(ii)* capital at time t is paid R_t, while the interest factor upon which consumers plan lifetime consumption and saving R_{t+1}, depends on future capital stock unknown at time t (timing problem); *(iii)* at time $t = 0$, k_0 is the already installed capital stock, but at time $t \geq 1$, k_t is the productive capital stock built from saving.

Individual constraints are $c_t^1 + s_t = w_t$ and $c_{t+1}^2 = R_{t+1} s_t$. The former states that individuals can decide how to allocate income between current consumption and saving; the latter states that during retirement agents dis-save completely: any kind of bequest is ruled out. Agents will optimally choose the consumption plan $\{c_t^1, c_{t+1}^2\}$, solving the following problem:
$\max\limits_{c_t^1, c_{t+1}^2} u(c_t^1 - \theta e_t) + v(c_{t+1}^2 - \delta h_{t+1})$ s.t. the individual constraints, $h_{t+1} = c_t^1$ and given the external stocks e_t and e_{t+1}.

[11] Using the *survival consumption* representation, $(c_t^1 - \theta e_t) > 0$ and $(c_{t+1}^2 - \delta h_{t+1})$,
$v_h = -\delta v_{c^2}$
$v_{hh} = \delta^2 v_{c^2 c^2}$
$v_{c^2 h} = -\delta v_{c^2 c^2}$.

First order conditions lead to the following Euler equation:

$$u_{c^1} - \delta v_{c^2} = R_{t+1} v_{c^2} \tag{1}$$

With respect to the standard Diamond's model (1956) in which $\delta = 0$, marginal utility of the young is lower, as we set $v_{c^2} > 0$: in order to achieve the same level of satisfaction when old, individuals need to correct young satisfaction by the (negative) habit effect.

The saving function is defined as $s_t = s(w_t, R_{t+1}, e_t)$, which is independent of h_{t+1} since the constraint $h_{t+1} = c_t^1$ is under agents' control, but it depends on e_t since $e_t = c_{t-1}^1$ is modelled as an externality, by assumption. Partial derivatives of the saving function are[12]

$$s_w = \frac{u_{c^1c^1} + \delta(R_{t+1}+\delta) v_{c^2c^2}}{u_{c^1c^1} + (R_{t+1}+\delta)^2 v_{c^2c^2}} > 0, \; s_r = \frac{-\left[v_{c^2} + v_{c^2c^2} c_{t+1}^2 \left(1 + \frac{\delta}{R_{t+1}}\right)\right]}{u_{c^1c^1} + (R_{t+1}+\delta)^2 v_{c^2c^2}}$$
$$s_e = \frac{-\theta u_{c^1c^1}}{u_{c^1c^1} + (R_{t+1}+\delta)^2 v_{c^2c^2}} < 0$$

The effect of the real wage is positive since an increase in real wage increases both current consumption and saving for retirement; the effect of the interest rate is ambiguous since it depends on the intertemporal elasticity of substitution; the effect of consumption externality is negative: an increase in e implies an increase in desire of current consumption and therefore, *ceteris paribus*, a reduction in saving. Comparing these derivatives with those by de la Croix and Michel (1999), we can conclude that they are generally smaller since they capture the habit effect through the extra term $-\delta v_{c^2}$ from equation (1).

Capital market equilibrium is given by the condition

$$k_{t+1} = s_t \tag{2}$$

Under assumptions on utility function and production function, the intertemporal equilibrium exists and it is unique.[13] The intertemporal equilibrium is k_0 and e_0 and a sequence $\{k_t, e_t\}_{t=1}^{\infty}$ following from individual constraints, production factors' equilibrium, saving function and market clearing (2), under the condition $e_t = c_{t-1}^1$:

$$k_{t+1} = s\left(f(k_t) - k_t f_k(k_t), f_k(k_{t+1}), e_t\right) \tag{3a}$$
$$e_{t+1} = f(k_t) - k_t f_k(k_t) - s\left(f(k_t) - k_t f_k(k_t), f_k(k_{t+1}), e_t\right) \tag{3b}$$

In the short-run, given the initial stock of capital and externality, k_0 and e_0, intragenerational non-separability increases saving of the adult ($\mathrm{d}s(k_0, e_0)/\mathrm{d}\delta > 0$), while intergenerational non-separability reduces it ($\mathrm{d}s(k_0, e_0)/\mathrm{d}\theta < 0$):[14]

$$\frac{\mathrm{d}s(k_0)}{\mathrm{d}\delta} = -\frac{v_{c^2}\left(c_1^2, h_1\right) + \left[f_k(k_1) + \delta\right] v_{c^2c^2}\left(c_1^2, h_1\right)\left(-w_0 + s_0\right)}{u_{c^1c^1}\left(c_0^1, e_0\right) + \left[f_k(k_1) + \delta\right]^2 v_{c^2c^2}\left(c_1^2, h_1\right)} > 0$$

[12] A proof is provided in the mathematical Appendix A.1.

[13] Galor and Ryder (1989) and de la Croix and Michel (2002), pp.20-27, provide a detailed proof for existence and uniqueness of the intertemporal equilibrium.

[14] See mathematical Appendix A.2.

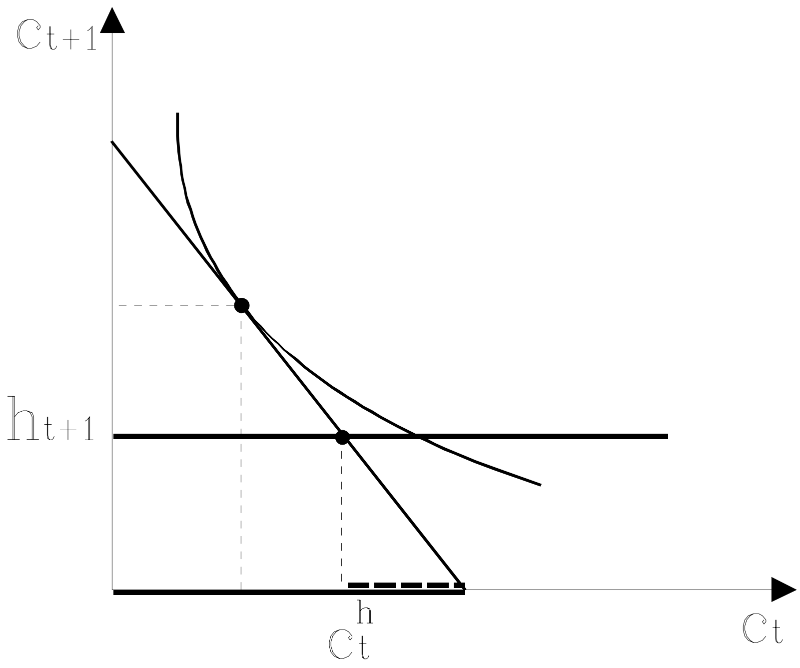

Figure 1: The effect of addiction

as $(-w_0 + s_0) < 0$ and

$$\frac{\mathrm{d}s\left(k_0\right)}{\mathrm{d}\theta} = -\frac{-u_{c^1c^1}\left(c_0^1, e_0\right)\left(-e_0\right)}{u_{c^1c^1}\left(c_0^1, e_0\right) + \left[f_k\left(k_1\right) + \delta\right]^2 v_{c^2c^2}\left(c_1^2, h_1\right)} < 0$$

as $e_0 > 0$.

An increase in δ implies an increase in the slope of the indifference curve, since it affects v_{c^2}. As a consequence, future marginal utility is higher ("distaste" effect) and consumption profile is steeper. Rising the weight of non-separable preferences across periods of life increases desire for consumption when old. In other words, to achieve the same level of satisfaction in terms of utility, the consumer has to re-schedule the consumption-saving plan in favour of saving. This explains why current consumption puts a floor for future consumption, under which agents do not want to go. Future consumption c_{t+1} is greater than future habit stock h_{t+1} only if current consumption c_t is lower than c_t^h, which is the level of current consumption associated to the level of future habits. In other words, current possibility of consumption is reduced in favour of future consumption. In Figure 1, potential consumption is drawn with a solid line and loss in consumption with a dashed line.

On the other hand, an increase in θ reduces the slope of the indifference curve, since it affects u_{c^1}. Consequently, current marginal utility is higher and

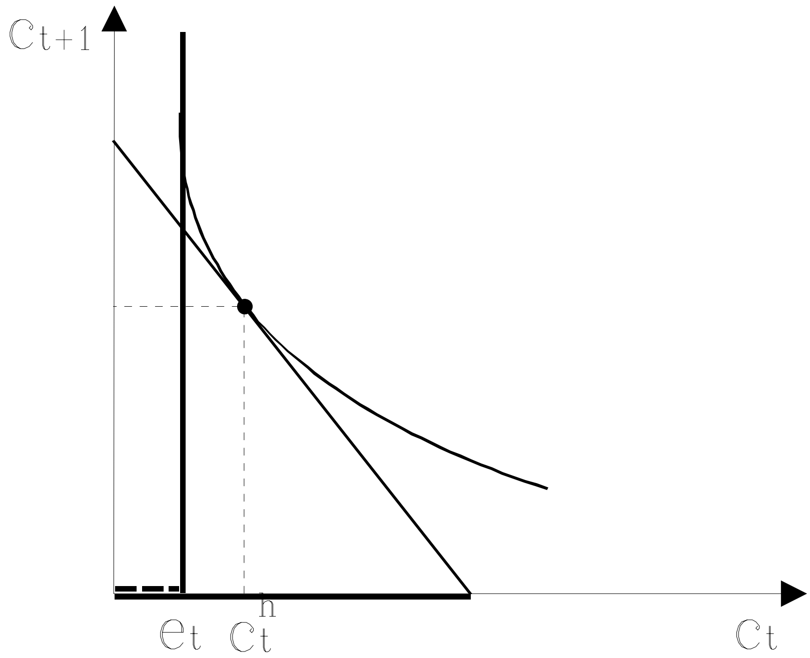

Figure 2: The effect of envy

the consumption profile is flatter.[15] This explains why "catching up with the Joneses" leads to higher consumption.[16] Under the assumption that starvation is ruled out, income w_t is greater than saving s_t. Since the intergenerational externality puts a floor on current consumption, therefore it reduces the ability to save of the young. In Figure 2 loss in saving is drawn with a dashed line and potential saving with a solid line.

The steady state equilibrium associated to the intertemporal equilibrium (3) is a combination of $k > 0$ and $e > 0$ such that

$$k = s\left(f\left(k\right) - kf_k\left(k\right), f_k\left(k\right), e\right) \quad (4a)$$
$$e = f\left(k\right) - kf_k\left(k\right) - s\left(f\left(k\right) - kf_k\left(k\right), f_k\left(k\right), e\right) \quad (4b)$$

Stability of the steady state (4) depends on parameters θ and δ.[17] As high-

[15] An alternative interpretation of the above result is the following: when β is higher than $(R + \delta)^{-1}$, individuals get lower returns if they anticipate consumption. Consequently, savings reduce. Note that this interpretation is independent of assumptions of addiction to habits and envy.

[16] Remind that the intergenerational externality cannot be modelled as a function of income, as the community income is non-observable directly. On the other hand, consumer goods are observable and thus aggregate consumption represent a reasonable tool to make comparison.

[17] For existence and uniqueness of the steady state, see Galor and Ryder (1989) and de la Croix and Michel (2002).

lighted by de la Croix and Michel (1999), for some values of these parameters hyperbolicity condition may not be satisfied. Therefore, we need to find critical value of θ and δ, namely $\hat{\theta}$ and $\hat{\delta}$, at which trajectories' change takes place and fixed point becomes non-hyperbolic. Parameter's value at which such a change occurs is called bifurcation. Non-hyperbolicity may arise when $\operatorname{mod} \sigma \left(\hat{\delta}, \hat{\theta}\right) = 1$ if the roots of the characteristic polynomial are real or when $r = 1$ if the roots are complex. If one of these conditions is met, linear approximation cannot be used to determine the stability of the system (Hartman-Grobman Theorem). Otherwise, local stability properties of the linear approximation carry over to the non-linear system.[18]

Under the assumptions that *(i)* $s_e = \frac{1-s_r f_{kk}}{k f_{kk}}$, *(ii)* $s_r > 1/f_{kk}$ and *(iii)* $\frac{|s_e|}{1-s_r f_{kk}} < 1 + \left[1 - \frac{s_w k |f_{kk}|}{1-s_r f_{kk}}\right]$, we find that the determinant is positive and equal to 1, the trace is positive and smaller than 2, the discriminant of the Jacobian matrix is negative and therefore that the eigenvalues are complex conjugate. Consequently, the fixed point is unstable and the bifurcation is identified by $\left(\hat{\delta}, \hat{\theta}\right)$ which are the roots of $\det \mathbf{J} = 1$.

In the long-run, intragenerational non-separability has a positive effect on steady state capital intensity, while intergenerational non-separability has a negative effect:[19]

$$
\begin{aligned}
\frac{\mathrm{d}k}{\mathrm{d}\delta} &= \frac{-v_{c^2} + [f_k(k) + \delta] v_{c^2c^2} c}{-[u_{c^1c^1} + u_{c^1e}]\eta^1 + f_{kk}(k) v_{c^2} + [f_k(k) + \delta] v_{c^2c^2}\{\eta^2\}} > 0 \\
\frac{\mathrm{d}k}{\mathrm{d}\theta} &= \frac{-e u_{c^1c^1}}{-[u_{c^1c^1} + u_{c^1e}]\eta^1 + f_{kk}(k) v_{c^2} + [f_k(k) + \delta] v_{c^2c^2}\{\eta^2\}} < 0
\end{aligned}
$$

$\eta^1 \equiv -k f_{kk}(k) - 1$
$\eta^2 \equiv [k f_{kk}(k) + f_k(k)] - \delta [-k f_{kk}(k) - 1]$
This result is a direct consequence of short-run effects. In fact, an increase in the addiction-to-habits parameter implies a steeper consumption profile, and therefore an increase in saving. As the equilibrium condition (2) holds, in the steady state an increase in saving causes an increase in capital stock. On the other hand, an increase in the externality parameter implies a flatter consumption profile and therefore a reduction in saving. Consequently, under steady state condition (4) an increase in saving causes a reduction in capital stock. We can therefore conclude that the model presents some cycles.[20]

A comparison between these opposite effects on steady state capital intensity

[18] See Appendix A.3 for mathematical details on the linear approximation of the dynamic system (3).

[19] See Appendix A.4.

[20] The fact that the present economy behaves as the one modelled by de la Croix and Michel (1999) should not surprise, as households exclusively control the intergenerational non-separability while the intragenerational non-separability is out of control Thus, the competitive equilibrium still displays fluctuations, but the bifurcation occurs in correspondence to different critical values of δ and θ.

is necessary at this point:

$$\Delta \equiv \left| \frac{\mathrm{d}k}{\mathrm{d}\delta} - \frac{\mathrm{d}k}{\mathrm{d}\theta} \right| = \left| \frac{-v_{c^2} + [f_k(k) + \delta] v_{c^2 c^2} c + e u_{c^1 c^1}}{-[u_{c^1 c^1} + u_{c^1 e}] \gamma^1 + f_{kk}(k) v_{c^2} + [f_k(k) + \delta] v_{c^2 c^2} \{\gamma^2\}} \right| > 0$$

The effect due to addiction is stronger than the effect due to envy, as habits are under agents' control, while the externality cannot be modified. As the presence of envy introduces here an intergenerational externality, «implying that the decentralized equilibrium is sub-optimal compared to the equilibrium that would maximize the planner's utility» (de la Croix and Michel (1999), p.521), in the next section we focus on the social planner's optimisation problem and we compare social optimum with competitive equilibrium.

4 Social optimum

The social planner's goal is to choose the optimal allocation that maximises social welfare. In particular, given the discount factor γ, the social planner chooses $\{c_t^1, c_t^2\}$ and $\{k_t, e_t\}$ such that
$\max\limits_{c_t^1, c_t^2} \sum\limits_{t=0}^{\infty} \gamma^t \left[u\left(c_t^1, e_t\right) + \frac{1}{\gamma} v\left(c_t^2, h_t\right) \right]$ s.t. resources constraint, $e_t = c_{t-1}^1$ and $h_t = c_{t-1}^1$ and given e_0 and k_0.
First order conditions are:

$$u_{c^1}\left(c_t^1, e_t\right) + \gamma u_e\left(c_{t+1}^1, e_{t+1}\right) = \frac{1}{\gamma} v_{c^2}\left(c_t^2, h_t\right) - v_h\left(c_{t+1}^2, h_{t+1}\right) \quad \text{(5a)}$$

$$\frac{1}{\gamma} v_{c^2}\left(c_t^2, h_t\right) = v_{c^2}\left(c_{t+1}^2, h_{t+1}\right) f_k\left(k_{t+1}\right) \quad \text{(5b)}$$

Equation (5a) represents the optimal intergenerational allocation of consumption between young and adult, in which the social planner internalises the externality associated to e. Marginal utility of the young, corrected for the externality through the additional term γu_e, is equated to the marginal utility of the old. Note that this social planner's first order condition does not respect equation (1). Moreover, with respect to the standard Diamond's model (1956) in which $\delta = \theta = 0$, marginal utility of the young, $u_{c^1}\left(c_t^1, e_t\right) + \gamma u_e\left(c_{t+1}^1, e_{t+1}\right)$, is lower, as we set $u_e < 0$, while marginal utility of the old, $\frac{1}{\gamma} v_{c^2}\left(c_t^2, h_t\right) - v_h\left(c_{t+1}^2, h_{t+1}\right)$, is higher, as we set $v_h < 0$. Equation (5b) sets the optimal intertemporal allocation.

The intertemporal equilibrium is thus defined as a sequence of $\left\{c_t^1\right\}_{t=1}^{\infty}$, $\left\{c_t^2\right\}_{t=1}^{\infty}$, $\{k_t\}_{t=1}^{\infty}$, $\{e_t\}_{t=1}^{\infty}$ following from first order conditions and from resources constraint, under the condition $e_t = h_t = c_{t-1}^1$:

$$\begin{array}{l} u_{c^1}\left(c_t^1, e_t\right) + \gamma u_e\left(c_{t+1}^1, e_{t+1}\right) = \frac{1}{\gamma} v_{c^2}\left(c_t^2, h_t\right) - v_h\left(c_{t+1}^2, h_{t+1}\right) \\ \frac{1}{\gamma} v_{c^2}\left(c_t^2, h_t\right) = v_{c^2}\left(c_{t+1}^2, h_{t+1}\right) f_k\left(k_{t+1}\right) \\ e_t = h_t = c_{t-1}^1 \Longrightarrow e_{t+1} = h_{t+1} = c_t^1 \Longrightarrow \left\{ \begin{array}{l} \mathrm{d}e_{t+1} = \mathrm{d}h_{t+1} = \mathrm{d}c_t^1 \\ \mathrm{d}e_t = \mathrm{d}h_t \end{array} \right. \\ k_{t+1} = f\left(k_t\right) - c_t^1 - c_t^2 \end{array} \quad (6)$$

Note that the social planner not only internalises the externality e, but also does not make any difference between e and h. The associated steady state is defined as a set $\left\{c^1 > 0, c^2 > 0, k > 0, e > 0\right\}$ such that the following constraints are satisfied

$$
\begin{aligned}
u_{c^1}\left(c^1, e\right) + \gamma u_e\left(c^1, e\right) &= \frac{1}{\gamma} v_{c^2}\left(c^2, h\right) - v_h\left(c^2, h\right) && \text{(7a)}\\
f_k(k) &= \frac{1}{\gamma} && \text{(7b)}\\
e &= h = c^1 && \text{(7c)}\\
f(k) &= c^1 + c^2 + k && \text{(7d)}
\end{aligned}
$$

Equation (7a) shows that in an economy with intra- *and* intergeneration non-separable preferences, the marginal utility of the young is lower than the corresponding marginal utility in Diamond (1956): "social" envy represents a benchmark from which individuals want to depart. Even marginal utility of the old is higher than the corresponding marginal utility in Diamond (1956): the same interpretation carries on. This means that, once we internalise the externality associated to the envy, intergenerational non-separability affects marginal utility in the same way as intragenerational non-separability and they both induce consumers to save. Equation (7b) is the modified golden rule, which is unchanged with tastes affected by the externality e. Equations (7c) and (7d) have been already discussed in the paper.

Proposition 1 (Stability of the steady state) *Under the assumption that capital stock and externality are state variables and that young and adult consumption are jump variables, locally explosive dynamics are possible, depending on the sign of the trace and of the element Z of the Jacobian matrix. If $\Delta \geq 0$, the four eigenvalues are real and local dynamics are either explosive or monotonic. If $\Delta < 0$, the eigenvalues are complex conjugate and local dynamics display either explosive or damped oscillation.*[21,22]

Proof. See Appendix B.2. ■

Let us give an economic interpretation of this result. Suppose that the initial level of k is low while the initial level of e is high. An individual may decide to reduce the level of current consumption, so that capital may fall down. Due to the assumption on intragenerational non-separability, this reduction in current consumption permits accumulation of capital through an increase in saving. As capital grows, it reaches the modified golden rule level but the economy is so used to low consumption that consumption grows slowly. As consumption grows, capital stops growing and begins falling. But as capital falls, current consumption falls as well. On the other hand, due to the assumption on intergenerational non-separability, this reduction in current consumption discourages

[21] See Appendix B.1 for mathematical details on the linear approximation of the dynamic system (6).

[22] Sufficient conditions for existence and uniqueness of the steady state are given by de la Croix and Michel (2002), p.527-528.

accumulation of capital through a reduction in saving. As capital falls, it moves away from the modified golden rule level. However, as capital falls, current consumption falls as well. Therefore, whatever the strenght of non-separability may operate, the final effect on consumption is always the same.

The above proposition states the possible patterns of optimal motion about the stationay point. The motion will converge to a modified golden rule if we have two stable roots. The motion will take the form of a saddle point when both the trace and the element Z are negative, of a node if the roots are real or of a focus if they are complex. The motion will diverge if we do not have two stable roots. The motion will take the form of a node if the roots are rela or of a focus if they are complex. Therefore, stability of the stationary point depends on the strenght of across-generations and across-periods interdependence, which affects the sign of the trace and of the element Z. Depending on the parameters δ and θ, the economy may converge to the steady state or diverge from it. In the numerical example presented in the next section, we will show that if $\delta \to 0$ and $\theta \to 1$ the economy diverges, while if $\delta \to 1$ and $\theta \to 0$ the economy converges to the steady state.

Our result crucially depends on the assumption of inter- and intragenerational non-separability of preferences. In fact, if the utility function is separable *both* across generations and across periods of life, the optimal solution in the one-sector overlapping generations model is generally characterized by monotonic convergence. If the utility function is non-separable across periods of life, Michel and Venditti (1997) show that even though eigenvalues of the system are real, convergence is not ensured any longer, so that endogenous cycles could appear. If the utility is non-separable across generations, but still separable across periods of life, de la Croix and Michel (1999) «show that (locally) converging oscillatory dynamics are highly likely» (de la Croix and Michel (1999), p.521). Finally, if the function is non-separable across periods of life *and* across generations, we show that explosive dynamics are possible.

Note that, contrary to what de la Croix and Michel (1999) state, the assumption of non-separability across periods of life *cannot* be modelled as a *simplifying hypothesis*, as it is a necessary and sufficient condition for the optimal solution to be characterised by explosive dynamics. As for the effect on steady state capital intensity, the effect of non-separable preferences across periods of life cumulates with the effect of non-separable preferences across generations "feeding" the cycle and therefore keeping economy towards explosion.[23]

5 A numerical example

A numerical example should help to illustrate the steady state behaviour of the economy in the competitive and optimal equilibrium. Taking a Cobb-Douglas production function $y_t = k_t^{\alpha}$ and a utility function of the form $\ln\left(c_t^1 - \theta e_t\right) + \beta \ln\left(c_{t+1}^2 - \delta c_t^1\right)$, we assign the following parameter values: $\alpha = 0.25$, $\theta =$

[23] For a discussion on the decentralization of the first best solution, we suggest to read de la Croix and Michel (1999), section 5.

Optimal economy (γ=0.5)	0.685±0.107i	2.851±.0446i
Optimal economy (γ=0.99)	0.536±0.179i	1.695±0.568i
Competitive economy	0.379±0.250i	

Figure 3: Eigenvalues

0/0.65, $\delta = 0/0.3$, $\beta = 0.43$ and two different planner's discount, $\gamma = 0.99$ and $\gamma = 0.5$. Values for α, θ, β and γ are taken from de la Croix and Michel (1999), while values for δ are taken from Wendner (2003)

Given the assigned parameters' values, steady state output is equal to 0.628 in the competitive economy.[24] In general, consumption of the young is higher than consumption of the old (0.316 versus 0.157). This result should not surprise as agents do not internalise the intergenerational externality e affecting their own consumption $\left(c^1 - \theta e\right)$ and therefore tend to consume more than what is optimal.

The eigenvalues presented in Figure 3 confirm that the competitive economy is characterised by endogenous cycle, depending on the parameters δ and θ. In particular the bifurcation is identified by $\hat{\delta} = 0.510$ and $\hat{\theta} = 0.5953^{-3}$, which are the roots of

$$1 - \frac{\alpha(1-\alpha)}{(1+\beta)}(\Phi + \Xi)\left\{\frac{\delta(1-\alpha)(\Phi+\Xi)}{[\alpha(\Phi+\Xi)+\delta]^2} + \beta\theta\right\} = 0$$

and it is represented in Figure 4. On the other hand, the optimal economy is characterised by local instability. Given the assigned values for the parameters, the trace and the element Z of the Jacobian matrix are both positive (4.483 and 7.228, respectively when $\gamma = 0.5$, and 4.462 and 7.148, respectively when $\gamma = 0.99$) and the four complex and conjugate eigenvalues identify an unstable focus: the oscillatory dynamics are explosive.[25]

[24] Appendix C provides algebric solutions for steady state values of all variables, when the utility function is logarithmic and the production function is Cobb-Douglas.

[25] See Appendix B.2 for further details on the stability of the optimal equilibrium.

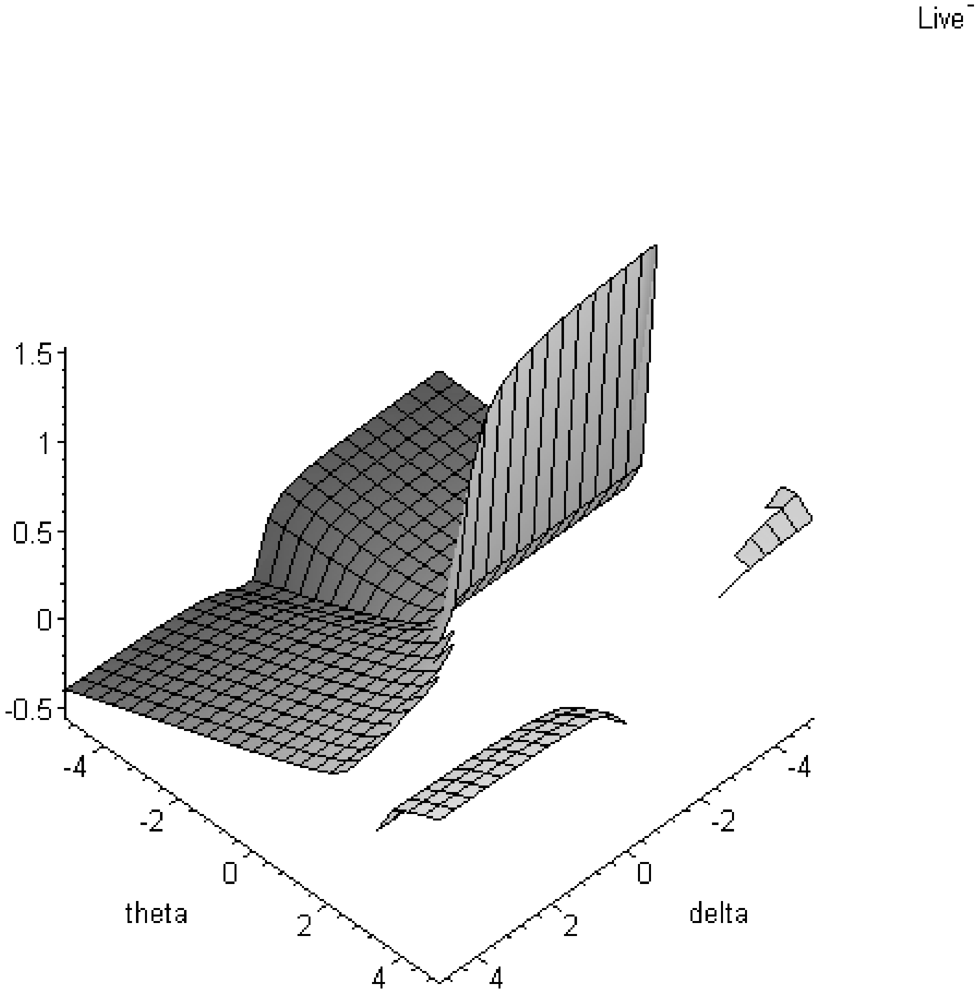

Figure 4: The bifurcation

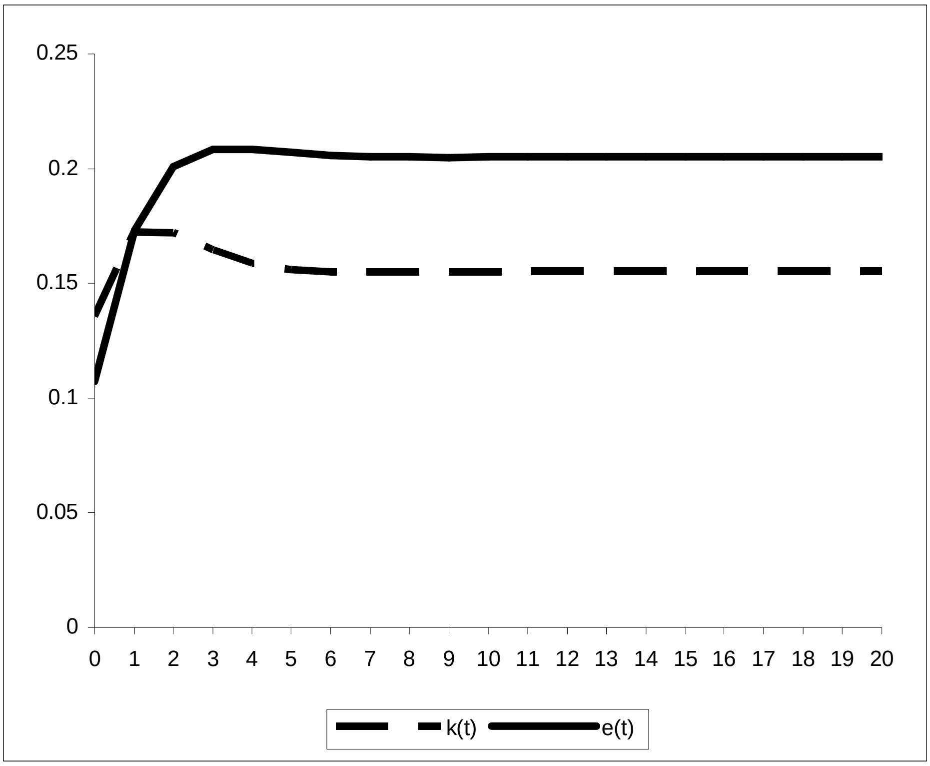

Figure 5: Steady state values of k and e

Starting with a negligible initial capital stock and low habits and externality, equilibrium of the competitive and optimal economy are presented in Figures 5 to 8, for two different values of γ. In particular, the competitive economy converges to the steady state values, $k = 0.155$ and $e = 0.205$ (Figure 5), and its dynamics is represented by a limit cycle (Figure 6). On the other hand, the optimal economy diverges, with a speed that grows with the discount parameter γ (Figure 7): when $\gamma = 0.5$, $k \to +\infty$ and $e \to -\infty$; when $\gamma = 0.99$, $k \to -\infty$ and $e \to +\infty$. Therefore, as the dynamics of the optimal economy is represented by a diverging trajectory to $+\infty$ when $\gamma = 0.99$ and to $-\infty$ when $\gamma = 0.5$ (Figure 8), we can conclude that the externality is the diminating strenght of the optimal economy.

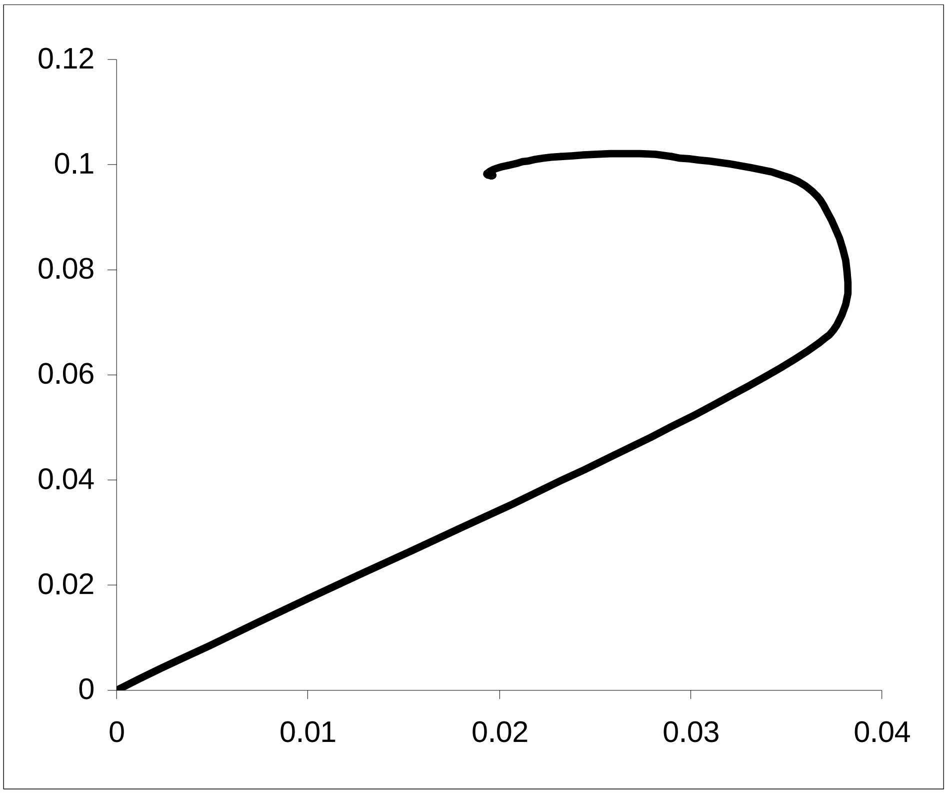

Figure 6: Limit cycle

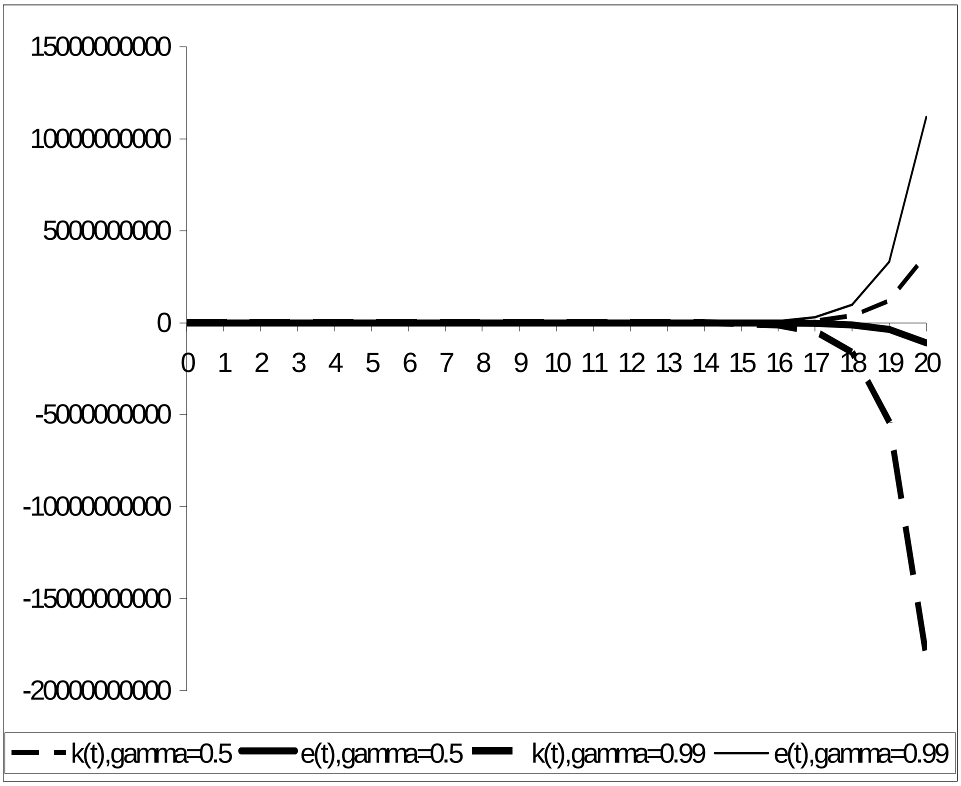

Figure 7: Optimal trajectories of k and e

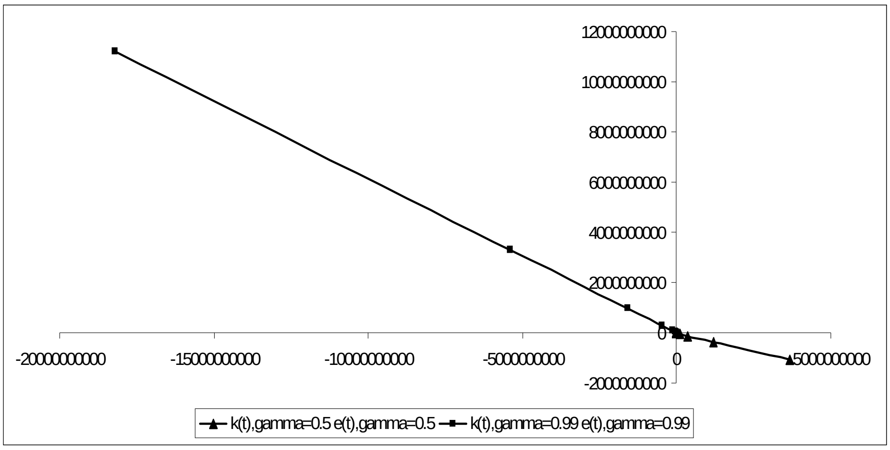

Figure 8: Dynamics of the optimal economy

6 Concluding remarks

The paper contributes to the literature on non-separable preferences. It points out that the introduction of intergenerational externalities together with intragenerational non-separability have important effects on optimal growth, transitional dynamics and stability properties of the equilibrium.

In the paper we extended an overlapping-generations model *a là* Diamond (1956), to allow for non-separable preferences.

In this framework, we address the issue of optimal growth. Studying the dynamic system associated to the social optimum, we show that

1. the competitive equilibrium displays fluctuations and therefore that convergence is not ensured (as in de la Croix and Michel (1999));

2. locally explosive dynamics are possible and that the optimal solution may *not* display damped oscillations (contrary to de la Croix and Michel (1999)).

The relevance of this result is that the assumption of separability across periods of life cannot be used as a simplifying device, because it affects dynamics and convergence to the steady state. Stability and dynamics of the model are highly dependent on the assumptions on non-separability of the utility function across periods of life *and* across generations. This result reflect the fact that, when we use a simplifying assumption, we have to pay attention to its consequences, as it could invalidate consolidated results.

An interesting extension of this model would be to introduce an exogenous probability of death p. In this case, the probability of death is binding because it has an influence on the economy: each old generation could restrain herself to consume all their saving and to re-schedule the consumption-saving plan in favour of youth consumption. Hence, given the consumption externality, the overall effect would be an increase in welfare of the young.

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A On the competitive equilibrium characteristics

A.1 Partial derivatives of the saving function

First, we re-write the maximisation programme as follows:

$$\max_{s_t} u\left(w_t - s_t - \theta e_t\right) + v\left(R_{t+1}s_t - \delta h_{t+1}\right) \text{ subject to } h_{t+1} = c_t^1 = w_t - s_t$$

The first order condition is

$$\text{FOC:} \qquad -u_{c^1} + \left(R_{t+1} + \delta\right) v_{c^2} = 0 \tag{A1}$$

which is nothing but an implicit function in s_t, w_t, R_{t+1} and e_t. To study the effect of a change in w_t, R_{t+1} or e_t on s_t, we need to apply the implicit function theorem. Therefore:

$$s_x = -\frac{\frac{\partial FOC}{\partial x}}{\frac{\partial FOC}{\partial s}} \tag{A2}$$

where we use x to identify any of the four variables which we want to study the effect of. First, we compute the denominator since it common to all the partial derivatives:

$$\frac{\partial FOC}{\partial s} = u_{c^1c^1} + \left(R_{t+1} + \delta\right)^2 v_{c^2c^2} \tag{A3}$$

given the assumptions on the utility function. Next, we move to the numerator:

$$\begin{aligned}
\frac{\partial FOC}{\partial w} &= -u_{c^1c^1} - \delta\left(R_{t+1} + \delta\right) v_{c^2c^2} \\
\frac{\partial FOC}{\partial r} &= v_{c^2} + \left(R_{t+1} + \delta\right) v_{c^2c^2} s_t \\
\frac{\partial FOC}{\partial e} &= -u_{c^1e} = \theta u_{c^1c^1}
\end{aligned}$$

Combining the above derivatives together with (A3) into the general equation (A2), we get:

$$\begin{aligned}
s_w &= -\frac{-u_{c^1c^1} - \delta\left(R_{t+1} + \delta\right) v_{c^2c^2}}{u_{c^1c^1} + \left(R_{t+1} + \delta\right)^2 v_{c^2c^2}} = \frac{u_{c^1c^1} + \delta\left(R_{t+1} + \delta\right) v_{c^2c^2}}{u_{c^1c^1} + \left(R_{t+1} + \delta\right)^2 v_{c^2c^2}} \\
s_r &= -\frac{v_{c^2} + \left(R_{t+1} + \delta\right) v_{c^2c^2} s_t}{u_{c^1c^1} + \left(R_{t+1} + \delta\right)^2 v_{c^2c^2}} = \frac{-\left[v_{c^2} + v_{c^2c^2} c_{t+1}^2 \left(1 + \frac{\delta}{R_{t+1}}\right)\right]}{u_{c^1c^1} + \left(R_{t+1} + \delta\right)^2 v_{c^2c^2}} \\
s_{e_t} &= -\frac{\theta u_{c^1c^1}}{u_{c^1c^1} + \left(R_{t+1} + \delta\right)^2 v_{c^2c^2}}
\end{aligned}$$

which are exactly the partial derivatives of the saving function.

A.2 Short-run effects on capital

In order to prove short-run effects on capital, we need to apply the implicit function theorem on the FOC (A1). In fact

$$s_\delta (k_0) = -\frac{\frac{\partial FOC}{\partial \delta}}{\frac{\partial FOC}{\partial s}}, \; s_\theta (k_0) = -\frac{\frac{\partial FOC}{\partial \theta}}{\frac{\partial FOC}{\partial s}}$$

where we evaluate the impact of the internal and external habit parameter in the initial capital stock k_0, since we focus on short-run effects. Therefore we compute

$$\frac{\partial FOC_0}{\partial \delta} = v_{c^2}\left(c_1^2, h_1\right) + \left[f_k\left(k_1\right) + \delta\right] v_{c^2c^2}\left(c_1^2, h_1\right)\left(-w_0 + s_0\right)$$

in which we use the market clearing condition for the interest rate, and

$$\frac{\partial FOC_0}{\partial \theta} = -u_{c^1c^1}\left(c_0^1, e_0\right)\left(-e_0\right)$$

For the denominator, we apply (A3), where $u_{c^1c^1}$ and $v_{c^2c^2}$ are evaluated in consumption, external and internal habits at time zero and 1:

$$\frac{\partial FOC_0}{\partial s} = u_{c^1c^1}\left(c_0^1, e_0\right) + \left[f_k\left(k_1\right) + \delta\right]^2 v_{c^2c^2}\left(c_1^2, h_1\right)$$

Thus we can conclude that

$$\begin{aligned}
\frac{\mathrm{d}s\left(k_0\right)}{\mathrm{d}\delta} &= -\frac{v_{c^2}\left(c_1^2, h_1\right) + \left[f_k\left(k_1\right) + \delta\right] v_{c^2c^2}\left(c_1^2, h_1\right)\left(-w_0 + s_0\right)}{u_{c^1c^1}\left(c_0^1, e_0\right) + \left[f_k\left(k_1\right) + \delta\right]^2 v_{c^2c^2}\left(c_1^2, h_1\right)} \\
\frac{\mathrm{d}s\left(k_0\right)}{\mathrm{d}\theta} &= -\frac{-u_{c^1c^1}\left(c_0^1, e_0\right)\left(-e_0\right)}{u_{c^1c^1}\left(c_0^1, e_0\right) + \left[f_k\left(k_1\right) + \delta\right]^2 v_{c^2c^2}\left(c_1^2, h_1\right)}
\end{aligned}$$

A.3 Linearization of the dynamic system (3)

First step: total differentiation of system (3) around the steady state (k, e) identified by the system (4)

$$\begin{aligned}
\mathrm{d}k_{t+1} &= s_w\left[-kf_{kk}\right]\mathrm{d}k_t + s_r f_{kk}\mathrm{d}k_{t+1} + s_e\mathrm{d}e_t \\
\mathrm{d}e_{t+1} &= -kf_{kk}\mathrm{d}k_t - s_w\left[-kf_{kk}\right]\mathrm{d}k_t - s_r f_{kk}\mathrm{d}k_{t+1} - s_e\mathrm{d}e_t
\end{aligned}$$

Second step: solve the first equation by $\mathrm{d}k_{t+1}$

$$\mathrm{d}k_{t+1} = \frac{1}{1 - s_r f_{kk}}\left[-s_w k f_{kk}\mathrm{d}k_t + s_e\mathrm{d}e_t\right]$$

and substitute it into the second

$$\mathrm{d}e_{t+1} = \frac{1}{1 - s_r f_{kk}}\left\{\left[s_w - 1 + s_r f_{kk}\right] k f_{kk}\mathrm{d}k_t - s_e\mathrm{d}e_t\right\}$$

Third step: re-write the linearised system in the matrix form:

$$\begin{bmatrix} \mathrm{d}k_{t+1} \\ \mathrm{d}e_{t+1} \end{bmatrix} = \frac{1}{1 - s_r f_{kk}}\begin{bmatrix} -s_w k f_{kk} & s_e \\ \left(s_w - 1 + s_r f_{kk}\right) k f_{kk} & -s_e \end{bmatrix}\begin{bmatrix} \mathrm{d}k_t \\ \mathrm{d}e_t \end{bmatrix}$$

A.4 Long-run effects on capital

We apply the implicit function theorem to FOC (1) evaluated in k, since we are studying the steady state of the economy. Since we already know the value of $\frac{\partial FOC}{\partial \delta}$ and $\frac{\partial FOC}{\partial \theta}$, we only differentiate the FOC (A1) with respect to k, which leads to

$$\begin{gathered} -\left[u_{c^1c^1}+u_{c^1e}\right]\left[-kf_{kk}\left(k\right)-1\right]+f_{kk}\left(k\right)v_{c^2}+ \\ \left[f_k\left(k\right)+\delta\right]v_{c^2c^2}\left\{\left[kf_{kk}\left(k\right)+f_k\left(k\right)\right]-\delta\left[-kf_{kk}\left(k\right)-1\right]\right\} \end{gathered}$$

Therefore, combining the computations we get

$$\begin{aligned} \frac{\mathrm{d}k}{\mathrm{d}\delta} &= -\frac{-v_{c^2}+\left[f_k(k)+\delta\right]v_{c^2c^2}c}{-\left[u_{c^1c^1}+u_{c^1e}\right]\left[-kf_{kk}(k)-1\right]+f_{kk}(k)v_{c^2}+\left[f_k(k)+\delta\right]v_{c^2c^2}\left\{\left[kf_{kk}(k)+f_k(k)\right]-\delta\left[-kf_{kk}(k)-1\right]\right\}} \\ \frac{\mathrm{d}k}{\mathrm{d}\theta} &= -\frac{eu_{c^1c^1}}{-\left[u_{c^1c^1}+u_{c^1e}\right]\left[-kf_{kk}(k)-1\right]+f_{kk}(k)v_{c^2}+\left[f_k(k)+\delta\right]v_{c^2c^2}\left\{\left[kf_{kk}(k)+f_k(k)\right]-\delta\left[-kf_{kk}(k)-1\right]\right\}} \end{aligned}$$

B On the optimal equilibrium characteristics

B.1 Linearization of the dynamic system (6)

First step: total differentiation of system (6)

$$\begin{aligned} &u_{c^1c^1}\mathrm{d}c_t^1+u_{c^1e}\mathrm{d}e_t+\gamma u_{c^1e}\mathrm{d}c_{t+1}^1+\gamma u_{ee}\mathrm{d}e_{t+1}=\tfrac{1}{\gamma}v_{c^2c^2}\mathrm{d}c_t^2+\tfrac{1}{\gamma}v_{c^2h}\mathrm{d}h_t-v_{c^2h}\mathrm{d}c_{t+1}^2-v_{hh}\mathrm{d}h_{t+1} \\ &\tfrac{1}{\gamma}v_{c^2c^2}\mathrm{d}c_t^2+\tfrac{1}{\gamma}v_{c^2h}\mathrm{d}h_t=v_{c^2c^2}f_k\left(k_{t+1}\right)\mathrm{d}c_{t+1}^2+v_{c^2h}f_k\left(k_{t+1}\right)\mathrm{d}h_{t+1}+v_{c^2}f_{kk}\left(k_{t+1}\right)\mathrm{d}k_{t+1} \\ &\mathrm{d}e_{t+1}=\mathrm{d}h_{t+1}=\mathrm{d}c_t^1 \\ &\mathrm{d}k_{t+1}=f_k\left(k_t\right)\mathrm{d}k_t-\mathrm{d}c_t^1-\mathrm{d}c_t^2 \end{aligned}$$

Second step: solve the first equation by $\mathrm{d}c_{t+1}^1$, second by $\mathrm{d}c_{t+1}^2$, using the third and the fourth, under the assumption that $\mathrm{d}h_t = \mathrm{d}e_t$:

$$\begin{aligned} \mathrm{d}c_{t+1}^1 &= \left[-\frac{u_{c^1c^1}}{\gamma u_{c^1e}}-\frac{\gamma u_{ee}}{\gamma u_{c^1e}}-\frac{v_{hh}}{\gamma u_{c^1e}}-\frac{v_{c^2h}}{\gamma u_{c^1e}}\left(\frac{v_{c^2}f_{kk}\left(k_{t+1}\right)}{v_{c^2c^2}f_k\left(k_{t+1}\right)}-\frac{v_{c^2h}}{v_{c^2c^2}}\right)\right]\mathrm{d}c_t^1+ \\ &\quad+\left[-\frac{1}{\gamma}+\frac{v_{c^2h}}{\gamma^2u_{c^1e}}-\frac{v_{c^2h}}{\gamma u_{c^1e}}\left(\frac{v_{c^2h}}{\gamma v_{c^2c^2}f_k\left(k_{t+1}\right)}\right)\right]\mathrm{d}e_t+ \\ &\quad+\left[\frac{v_{c^2c^2}}{\gamma^2u_{c^1e}}-\frac{v_{c^2h}}{\gamma u_{c^1e}}\left(\frac{1}{\gamma f_k\left(k_{t+1}\right)}+\frac{v_{c^2}f_{kk}\left(k_{t+1}\right)}{v_{c^2c^2}f_k\left(k_{t+1}\right)}\right)\right]\mathrm{d}c_t^2+ \\ &\quad+\frac{v_{c^2h}}{\gamma u_{c^1e}}\left[\frac{v_{c^2}f_{kk}\left(k_{t+1}\right)}{v_{c^2c^2}f_k\left(k_{t+1}\right)}f_k\left(k_t\right)\right]\mathrm{d}k_t \\ \mathrm{d}c_{t+1}^2 &= \left[\frac{v_{c^2}f_{kk}\left(k_{t+1}\right)}{v_{c^2c^2}f_k\left(k_{t+1}\right)}+\frac{1}{\gamma f_k\left(k_{t+1}\right)}\right]\mathrm{d}c_t^2+\frac{v_{c^2h}}{\gamma v_{c^2c^2}f_k\left(k_{t+1}\right)}\mathrm{d}e_t+ \\ &\quad-\frac{v_{c^2}f_{kk}\left(k_{t+1}\right)}{v_{c^2c^2}f_k\left(k_{t+1}\right)}f_k\left(k_t\right)\mathrm{d}k_t+\left[\frac{v_{c^2}f_{kk}\left(k_{t+1}\right)}{v_{c^2c^2}f_k\left(k_{t+1}\right)}-\frac{v_{c^2h}}{v_{c^2c^2}}\right]\mathrm{d}c_t^1 \\ \mathrm{d}e_{t+1} &= \mathrm{d}h_{t+1}=\mathrm{d}c_t^1 \\ \mathrm{d}k_{t+1} &= f_k\left(k_t\right)\mathrm{d}k_t-\mathrm{d}c_t^1-\mathrm{d}c_t^2 \end{aligned}$$

Third step: re-write the linearised system in the matrix form:

$$\begin{bmatrix} \mathrm{d}e_{t+1} \\ \mathrm{d}c^1_{t+1} \\ \mathrm{d}k_{t+1} \\ \mathrm{d}c^2_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \gamma^{-1}\left[B\left(1-\gamma E\right)-1\right] & A-B\left(D-E\right) & \gamma^{-1}DB & C-B\left(1+D\right) \\ 0 & -1 & \gamma^{-1} & -1 \\ E & D-E & -\gamma^{-1}D & 1+D \end{bmatrix} \begin{bmatrix} \mathrm{d}e_t \\ \mathrm{d}c^1_t \\ \mathrm{d}k_t \\ \mathrm{d}c^2_t \end{bmatrix}$$

where we set
$A \equiv -\frac{u_{c^1c^1}+\gamma u_{ee}+v_{hh}}{\gamma u_{c^1e}} > 0$
$B \equiv \frac{v_{c^2h}}{\gamma u_{c^1e}} > 0$
$C \equiv \frac{v_{c^2c^2}}{\gamma^2 u_{c^1e}} < 0$
$D \equiv \frac{\gamma v_{c^2} f_{kk}(k_{t+1})}{v_{c^2c^2}} > 0$
$E \equiv \frac{v_{c^2h}}{v_{c^2c^2}} < 0$
under the assumption that in steady state $f_k\left(k_{t+1}\right) = f_k\left(k_t\right) = \gamma^{-1}$. The determinant of the Jacobian matrix is $\det \mathbf{J} = \gamma^{-2}$, while the trace is $T_{\mathbf{J}} = 1+\frac{1}{\gamma}+A-B\left(D-E\right)+D$. Note that while the determinant is always positive, sign of the trace $T_{\mathbf{J}}$ depends on the sign of $\left\{1+\frac{1}{\gamma}+A-B\left(D-E\right)+D\right\}$.

B.2 Stability of the linearised system

In order to study the stability of the linearised system, we need to study the characteristic polynomial C in the eigenvalue σ:

$$C\left(\sigma\right) = \sigma^4 - T_{\mathbf{J}}\sigma^3 + Z\sigma^2 - \gamma^{-1}T_{\mathbf{J}}\sigma + \det\mathbf{J} = 0$$

where
$T_{\mathbf{J}} = 1+\frac{1}{\gamma}+A-B\left(D-E\right)+D \gtrless 0$ if $1+\frac{1}{\gamma}+A+D \gtrless B\left(D-E\right)$
$Z = \frac{2}{\gamma}+\left(A+BE\right)\left(1+\frac{1}{\gamma}\right)+\left(A-C\right)D \gtrless 0$ if $\frac{2}{\gamma}+\left(A-C\right)D \gtrless \left(1+\frac{1}{\gamma}\right)\left(A+BE\right)$
$\det\mathbf{J} = \gamma^{-2}\left(1-B+\gamma EC\right) = \gamma^{-2} > 0$.
Following de la Croix and Michel (1999), we can factorise the polynomial into

$$C\left(\sigma\right) = \left(\sigma-\sigma_1\right)\left(\sigma-\frac{1}{\gamma\sigma_1}\right)\left(\sigma-\sigma_2\right)\left(\sigma-\frac{1}{\gamma\sigma_2}\right) = 0$$

which is equivalent to

$$\left(\sigma^2-\phi_1\sigma+\frac{1}{\gamma}\right)\left(\sigma^2-\phi_2\sigma+\frac{1}{\gamma}\right) = 0$$

Now we consider different scenarios, due to the sign's ambiguity of the trace $T_{\mathbf{J}}$ and of the element Z. First, we assume that $T_{\mathbf{J}}$ and Z are both positive. Therefore, we consider two possible cases:

- $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geq 0$, $i = 1, 2$. The four eigenvalues are real and they can be:

(i) four negative. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$, and it is excluded as it violates $T_{\mathbf{J}} > 0$.
(ii) two negative and two positive. This case implies that $\phi_1 + \phi_2 \gtrless 0$ and $\phi_1 \cdot \phi_2 < 0$, and it is excluded as it violates $Z > 0$.
(iii) four positive. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$, and therefore it respects both conditions on the trace and on Z.

- $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0$, $i = 1, 2$. We look at the real parts. Since $a = -\frac{1}{2}\phi_i \neq 0$, $i = 1, 2$, the eigenvalues are complex and conjugate. Thus we distinguish three possible cases:
(i) four negative. This case implies $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$, and it is excluded as it violates condition on the trace $T_{\mathbf{J}}$.
(ii) two negative and two positive. This case implies that $\phi_1 + \phi_2 \gtrless 0$ and $\phi_1 \cdot \phi_2 < 0$, and it is excluded as it violates condition on Z.
(iii) four positive. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$, and therefore it respects both conditions on the trace and on Z.

Under the assumption that the trace and the element Z are both positive, the only admissible case *(iii)* identifies an unstable node, if the eigenvalues are real, and an unstable focus, if the eigenvalues are complex conjugate. Locally explosive dynamics are highly likely.

Second, we assume that $T_{\mathbf{J}}$ and Z are both negative. Again, we consider two possible cases:

- $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geq 0$, $i = 1, 2$. The four eigenvalues are real and they can be:
(i) four positive. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$, and therefore it does not respect both conditions on the trace and on Z.
(ii) two negative and two positive. This case implies that $\phi_1 + \phi_2 \gtrless 0$ and $\phi_1 \cdot \phi_2 < 0$ and it is an admissible case only if $\phi_1 + \phi_2 < 0$ as $T_{\mathbf{J}} < 0$.
(iii) four negative. This case implies that $\phi_1 \cdot \phi_2 > 0$ and $\phi_1 + \phi_2 < 0$ and therefore it is excluded as it does not respects condition on Z.

- $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0$, $i = 1, 2$. We look at the real parts first. Since $a = -\frac{1}{2}\phi_i \neq 0$, $i = 1, 2$, the eigenvalues are complex and conjugate. Thus we distinguish three possible cases:
(i) four positive. This case implies $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$, and therefore it does not respect both conditions on the trace and on Z.
(ii) two negative and two positive. This case implies that $\phi_1 + \phi_2 \gtrless 0$ and $\phi_1 \cdot \phi_2 < 0$ and it is an admissible case only if $\phi_1 + \phi_2 < 0$ as $T_{\mathbf{J}} < 0$.
(iii) four negative. This case implies that $\phi_1 \cdot \phi_2 > 0$ and $\phi_1 + \phi_2 < 0$ and therefore it is excluded as it does not respects condition on Z.

Under the assumption that the trace and the element Z are both negative, the only admissible case *(ii)* identifies a stable saddle point, that ensures monotonic local convergence.

Finally, we assume that $T_{\mathbf{J}}$ and Z have opposite sign. Again, we consider two possible cases:

- $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geq 0$, $i = 1, 2$. The four eigenvalues are real and they can be:
  (i) four positive. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$, and therefore it does not respect either conditions on the trace or on Z.
  (ii) two negative and two positive. This case implies that $\phi_1 + \phi_2 \gtrless 0$ and $\phi_1 \cdot \phi_2 < 0$ and it is an admissible case only if $\phi_1 + \phi_2 > 0$ as $T_{\mathbf{J}} > 0$.
  (iii) four negative. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$ and it is a possible case only if $T_{\mathbf{J}} < 0$ and $Z > 0$.

- $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0$, $i = 1, 2$. We look at the real parts first. Since $a = -\frac{1}{2}\phi_i \neq 0$, $i = 1, 2$, the eigenvalues are complex and conjugate. Thus we distinguish three possible cases:
  (i) four positive. This case implies $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$, and therefore it does not respect either conditions on the trace or on Z.
  (ii) two negative and two positive. This case implies that $\phi_1 + \phi_2 \gtrless 0$ and $\phi_1 \cdot \phi_2 < 0$. Therefore, it is admissible only if $\phi_1 + \phi_2 > 0$ as $T_{\mathbf{J}} > 0$.
  (iii) four negative. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$ and it is a possible case only if $T_{\mathbf{J}} < 0$ and $Z > 0$.

Under the assumption that the trace and the element Z have opposite sign, case *(ii)* identifies an unstable solution as it imposes $T_{\mathbf{J}} > 0$, and therefore locally explosive dynamics are highly likely. Case *(iii)* identifes a stable node for real eigenvalues and a stable focus for complex conjugate eigenvalues, and therefore it ensures damped convergence to the steady state.

C Steady state values of k and e

Under the assumption that

$$U = \ln\left(c_t^1 - \theta e_t\right) + \beta \ln\left(c_{t+1}^2 - \delta c_t^1\right)$$

and

$$y_t = k_t^\alpha$$

steady state equilibrium (4) becomes

$$\begin{aligned} k &= \frac{\beta}{1+\beta}\left[(1-\alpha)k^\alpha - \theta e\right] + \frac{\delta(1-\alpha)k^\alpha}{(\alpha k^{\alpha-1}+\delta)(1+\beta)} \qquad \text{(C1a)} \\ e &= \frac{\alpha(1-\alpha)k^{2\alpha-1}}{(\alpha k^{\alpha-1}+\delta)(1+\beta)} + \frac{\beta}{1+\beta}\theta e \end{aligned}$$

From the second we find out e:

$$e = \frac{\alpha(1-\alpha)}{[1+\beta(1-\theta)]}\left[\frac{k^{2\alpha-1}}{(\alpha k^{\alpha-1}+\delta)}\right] = \alpha'\left[\frac{k^{2\alpha-1}}{(\alpha k^{\alpha-1}+\delta)}\right] \qquad \text{(C2)}$$

where $\alpha' \equiv [\alpha(1-\alpha)] / [1+\beta(1-\theta)]$. Now we substitute e into the first equation and we find k:

$$\beta[\alpha'\theta - \alpha(1-\alpha)] k^{2(\alpha-1)} + (1+\beta)[\alpha - (1-\alpha)\delta] k^{\alpha-1} + (1+\beta)\delta = 0$$

Since $\alpha' \equiv [\alpha(1-\alpha)\mu] / [1+\beta(1-\mu\theta)]$,

$$\begin{aligned}
\beta[\alpha'\theta - \alpha(1-\alpha)] &= \beta\left[\frac{\alpha(1-\alpha)}{1+\beta(1-\theta)} - \alpha(1-\alpha)\right] = \\
&= \beta\left[\frac{\alpha(1-\alpha)\theta - \alpha(1-\alpha)(1+\beta-\beta\theta)}{1+\beta(1-\theta)}\right] = \\
&= \frac{\alpha(1-\alpha)\beta}{1+\beta(1-\theta)}[\theta - 1 - \beta + \beta\theta] = \\
&= -\frac{\alpha(1-\alpha)\beta(1+\beta)(1-\theta)}{1+\beta(1-\theta)}
\end{aligned}$$

and the above capital equation becomes

$$\begin{aligned}
-\frac{\alpha(1-\alpha)\beta(1+\beta)(1-\mu\theta)}{1+\beta(1-\mu\theta)} k^{2(\alpha-1)} + (1+\beta)[\alpha - (1-\alpha)\delta\lambda] k^{\alpha-1} + (1+\beta)\delta\lambda &= 0 \\
-\frac{\alpha(1-\alpha)\beta(1-\mu\theta)}{[1+\beta(1-\mu\theta)]\delta\lambda} k^{2(\alpha-1)} + \frac{[\alpha - (1-\alpha)\delta\lambda]}{\delta\lambda} k^{\alpha-1} + 1 &= 0
\end{aligned}$$

Set
$a \equiv -\frac{\alpha(1-\alpha)\beta(1-\theta)}{\delta[1+\beta(1-\theta)]}$
$b \equiv \frac{[\alpha-(1-\alpha)\delta]}{\delta}$
$c \equiv 1$
and
$k^{(\alpha-1)} \equiv x \Longrightarrow k^{2(\alpha-1)} \equiv x^2$. Thus

$$ax^2 + bx + c = 0 \tag{C3}$$

has the following set of possible solutions

$$\begin{aligned}
x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \\
&= \frac{-\frac{[\alpha-(1-\alpha)\delta]}{\delta} \pm \sqrt{\left\{\frac{[\alpha-(1-\alpha)\delta]}{\delta}\right\}^2 + 4\frac{\alpha(1-\alpha)\beta(1-\theta)}{\delta[1+\beta(1-\theta)]}}}{-2\frac{\alpha(1-\alpha)\beta(1-\theta)}{\delta[1+\beta(1-\theta)]}}
\end{aligned}$$

Since capital cannot be an imaginary number, the discriminant

$$\Delta \equiv \left\{\frac{[\alpha - (1-\alpha)\delta]}{\delta}\right\}^2 + 4\frac{\alpha(1-\alpha)\beta(1-\theta)}{\delta[1+\beta(1-\theta)]}$$

has to be non-negative. In particular, as $\alpha, \beta, \delta, (1-\theta), (1-\alpha), [1+\beta(1-\theta)] > 0$, we can conclude that the discriminant is strictly positive.

Now we proceed and study the sign of the sequence $\{a, b, c\}$. By Descartes' theorem, equation (C3) has a positive and a negative solution, whatever the sign of coefficient b is. Since capital is never negative by definition, we exclude the negative solution. Therefore, the unique solution for equation (C3) is

$$\begin{aligned} x &= \frac{[\alpha - (1-\alpha)\,\delta] + \sqrt{[\alpha - (1-\alpha)\,\delta]^2 + 4\frac{\alpha(1-\alpha)\beta(1-\theta)}{\delta[1+\beta(1-\theta)]}}}{2\frac{\alpha(1-\alpha)\beta(1-\theta)}{\delta[1+\beta(1-\theta)]}} = \\ &= \frac{1}{2}\left[\alpha - (1-\alpha)\,\delta\right]\left[\frac{1+\beta\,(1-\theta)}{\alpha\,(1-\alpha)\,\beta\,(1-\theta)}\right] + \\ &\quad + \frac{1}{2}\left[\frac{1+\beta\,(1-\theta)}{\alpha\,(1-\alpha)\,\beta\,(1-\theta)}\right]\sqrt{[\alpha - (1-\alpha)\,\delta]^2 + 4\frac{\alpha\,(1-\alpha)\,\beta\delta\,(1-\theta)}{[1+\beta\,(1-\theta)]}} \end{aligned}$$

Using $k^{(\alpha-1)} \equiv x$, the steady state values of k is

$$k = \left[\begin{array}{c} \frac{1}{2}\left[\alpha - (1-\alpha)\,\delta\right]\left[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}\right] + \\ + \frac{1}{2}\left[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}\right]\sqrt{[\alpha - (1-\alpha)\,\delta]^2 + 4\frac{\alpha(1-\alpha)\beta\delta(1-\theta)}{[1+\beta(1-\theta)]}} \end{array}\right]^{-\frac{1}{1-\alpha}} \tag{C4}$$

Therefore, steady state solution for the (C1) is given by equations (C2) and (C4):

$$\begin{aligned} k &= (\Phi + \Xi)^{-\frac{1}{1-\alpha}} \\ e &= \Omega\left[\frac{(\Phi + \Xi)^{\frac{1-2\alpha}{1-\alpha}}}{\alpha\,(\Phi + \Xi) + \delta}\right] \end{aligned}$$

where

$$\begin{aligned} \Phi &\equiv \frac{1}{2}\left[\alpha - (1-\alpha)\,\delta\right]\left[\frac{1+\beta\,(1-\theta)}{\alpha\,(1-\alpha)\,\beta\,(1-\theta)}\right] \\ \Xi &\equiv \frac{1}{2}\left[\frac{1+\beta\,(1-\theta)}{\alpha\,(1-\alpha)\,\beta\,(1-\theta)}\right]\sqrt{[\alpha - (1-\alpha)\,\delta]^2 + 4\frac{\alpha\,(1-\alpha)\,\beta\delta\,(1-\theta)}{[1+\beta\,(1-\theta)]}} \\ \Omega &\equiv \frac{\alpha\,(1-\alpha)\,\theta}{1+\beta\,(1-\theta)} \end{aligned}$$

Similarly, we can prove that, when the production function is Cobb-Douglas and the utility function is of the form $\ln\left(c_t^1 - \theta e_t\right) + \beta \ln\left(c_{t+1}^2 - \delta c_t^1\right)$, steady state equilibrium (7) becomes

$$\begin{aligned} \frac{1}{c^1 - \theta e} - \theta\gamma\frac{1}{c^1 - \theta e} &= \frac{1}{\gamma\,(c^2 - \delta h)} + \frac{\delta}{c^2 - \delta h} \\ \alpha k^{\alpha-1} &= \frac{1}{\gamma} \\ e &= h = c^1 \\ k^\alpha &= c^1 + c^2 + k \end{aligned}$$

Solving the first by c^2, we get

$$c^2 = \Psi c^1$$

where $\Psi \equiv \frac{\beta\left(\frac{1}{\gamma}+\delta\right)(1-\theta)}{1-\gamma\theta} + \delta$. We substitute this ewpression for c^2 into the resource constraint:

$$k^\alpha = (1+\Psi)\left[(1-\alpha)k^\alpha - k\right] + k$$

where we use the fact that in equilibrium $c^1 = (1-\alpha)k^\alpha - k$. Then, we solve

$$k^\alpha - (1+\Psi)\left[(1-\alpha)k^\alpha - k\right] - k = 0$$

by k and we get

$$k = \left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{1}{1-\alpha}}$$

Consequently,

$$\begin{aligned} e &= c = (1-\alpha)\left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{\alpha}{1-\alpha}} - \left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{1}{1-\alpha}} \\ d &= \Psi\left\{(1-\alpha)\left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{\alpha}{1-\alpha}} - \left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{1}{1-\alpha}}\right\} \end{aligned}$$