Endogenous fluctuations in a one capital good model

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Abstract

The present paper studies first-best solutions to the aggregative model of growth with externalities. Pareto-efficient accumulation paths are the result of policy measures that correct externalities. We show that there exist non-constant periodic solutions as first-best paths. In particular, we prove the occurrence of a Hopf bifurcation for optimal paths of the deterministic aggregative model as the result of policy changing investment behavior in a Pareto-efficient manner.

1. Introduction

An established result in the growth literature is that endogenous cycles cannot occur in efficient economies with one capital good. Multiple capital good economies can, however, be characterized by deterministic cycles as optimal growth paths, as demonstrated by Benhabib and Nishimura [1] , Medio [8], Benhabib and Rustichini [2] and Cartigny and Venditti [4].

The purpose of the present paper is to re-investigate the issue of equilibrium cycling of optimal paths resulting from the interaction between technological externalities and optimal policy. Externalities are corrected by means of non-linear taxes which are distinct from corrective pigouvian instruments. The obtained efficient paths are thus different from both command optima and solutions to the inverse optimal problem. As a consequence, they display peculiar local stability properties implying the existence of deterministic cycles of competitive one-capital good economies. To show the result, we apply Hopf theorem. In particular, the modified Hamiltonian dynamical system experiences a bifurcation from a steady solution into periodic solutions even for strictly positive values of the time preference rate. Hence the occurrence of closed orbits is not a result restricted to multi capital good models, but extends to one capital good economies. The result holds under the standard assumptions of strict concavity of the direct utility function and convexity of privates' technology.

The paper is organized as follows. Section 2 puts forth the model. Section 3 characterizes corrective policy. Section 4 studies the effects of policy intervention on local stability and proves the existence of closed orbits of one-capital good first-best competitive paths. Section 5 ends the paper with a summary of the results.

2. The model

Our objective is to study the dynamic effects of a policy intervention that takes into account the difference between the command optimum program a benevolent social planner would impose and the *laissez-faire* dynamics that are inefficient in the presence of technological externalities. Hence, we will focus on two couples of generalized sequences of state and adjoint variables.

The instantaneous indirect utility function

$$u : D \subset \overset{\circ}{R}_{+} \times R \rightarrow R$$

is C^2 and concave.

The social planner's problem consists of choosing $\dot{\hat{k}} : \left(\hat{k}, \dot{\hat{k}}\right) \in D$, where D is non-empty and convex, so as to maximize, at every instant, the Hamiltonian function

$$\hat{H}\left(p, \hat{k}\right) \triangleq u\left(\hat{k}, \dot{\hat{k}}\right) + p\dot{\hat{k}}$$

The associated command optimum is determined by the modified Hamiltonian dynamical system (MHDS)

$$\begin{aligned} \hat{H}_p &= \dot{\hat{k}} \\ \hat{H}_{\hat{k}} &= \delta p - \dot{p} \end{aligned} \tag{1}$$

where

$$\dot{\hat{k}} \triangleq f\left(\hat{k}, \hat{k}\right) - c\left(p\right)$$

is the command optimum investment equation, where aggregate capital enters the social production function $f\left(\hat{k}, \hat{k}\right)$ as a variable input and the planner receives the full social marginal productivity from the level of the aggregate capital stock. The generalized sequence $\left\{\hat{k}_t\right\}_{t\in[0,\infty]}$ is the trajectory of capital stocks chosen by the planner, and $\{p_t\}_{t\in[0,\infty]}$ are the associated prices. The transversality condition (TVC) requires

$$\lim_{t\to\infty} e^{-\delta t} p_t \hat{k}_t = 0$$

where the time preference rate δ is strictly positive.

On the other hand, the efficiently corrected investment by private agents, $\dot{k} : \left(k, \dot{k}\right) \in D$, maximizes the Hamiltonian

$$H\left(q, k\right) \triangleq u\left(k, \dot{k}\right) + q\dot{k}$$

and implies the competitive path

$$\begin{aligned} H_q &= \dot{k} \\ H_k &= \delta q - \dot{q} \end{aligned} \tag{2}$$

where

$$\dot{k} \triangleq f(k,K) - c(q) + T$$

is the sum of the *laissez-faire* investment equation

$$f(k,K) - c(q)$$

plus corrective policy intervention T, with $\frac{dT}{dk} = \tau$ such that $H_k = \hat{H}_k$.

Under *laissez-faire*, private firms are unable to affect the aggregate capital level. Hence, K enters the privates' production function $f(k,K)$ as fixed parameter. This implies the private marginal productivity (accruing to firms) differs from social productivity.

In the presence of corrective policy, the sequence of capital stocks is denoted by $\{k_t\}_{t\in[0,\infty]}$, while $\{q_t\}_{t\in[0,\infty]}$ is the sequence of market prices for investment. In this case, the TVC is

$$\lim_{t\to\infty} e^{-\delta t} p_t k_t = 0$$

3. Corrective policy

The corrective policy intervention T is such as to equate the rental rate for capital under *laissez-faire* to the command optimum value. With a negative technological externality, τ is a corrective non-linear tax. This is because each private firm recognizes the value of the private marginal product, but neglects the social value of the technological externality. As a consequence, consumption is too low and investment is too high. Corrective policy determines a reallocation of current goods away from investment and towards consumption which is welfare maximizing.

By equating H_k to $\hat{H}_{\hat{k}}$, it follows

Proposition 1. *The corrective tax is*

$$\tau \triangleq \frac{p}{q}\left(\frac{df\left(\hat{k},\hat{k}\right)}{d\hat{k}}\right) - \frac{df(k,K)}{dk} \tag{3}$$

Proof. >From

$$H_k = \hat{H}_{\hat{k}} \tag{4}$$

it follows

$$u_k + q\frac{df(k,K)}{dk} + q\tau = u_{\hat{k}} + p\frac{df\left(\hat{k},\hat{k}\right)}{d\hat{k}} \tag{5}$$

Then, for the same capital stocks, i.e. $k = \hat{k}$, we have

$$\tau = \frac{p}{q}\left(\frac{df\left(k,k\right)}{dk}\right) - \frac{df\left(k,K\right)}{dk}$$

■

The ratio $\frac{p}{q}$ is the relative price between command optimum investment and investment under *laissez-faire*. It can be interpreted as the rate of transformation between command optimum investment and market first-best investment at a point in time. Thus $\frac{p}{q}\left(\frac{df\left(\hat{k},\hat{k}\right)}{d\hat{k}}\right)$ is the market measure of social marginal productivity. It follows that τ is the value of the externality that market prices fail to reveal. Policy intervention transforms the *laissez-faire* model into a one-sector model with investment depending on the relative price $\frac{p}{q}$. In the particular case where $\frac{p}{q} = 1$, the model reduces to the optimal one-sector model. In this case the corrective policy is equivalent to the command optimum and the corrective tax is a linear pigouvian tax. Under fairly general conditions, this policy is optimal. Let t_0 denote the initial timing of policy adoption, then

Theorem 2. *Under Assumption 1, symmetric equilibrium,* $\frac{p_{t_0}}{q_{t_0}} < 1$, $T_0 = 0$, *the corrective policy* (3) *is optimal.*

Proof. Total welfare under *laissez-faire* is

$$W_{lf} \triangleq \delta \int_{t_0}^{\infty} u\left(c_t\right) e^{-\delta t} dt = u\left(c_{t_0}\right) + q_{t_0}\left[f\left(k_{t_0}, K_{t_0}\right) - c\left(q_{t_0}\right)\right]$$

and welfare under command optimum is

$$W_{co} \triangleq \delta \int_{t_0}^{\infty} u\left(\hat{c}_t\right) e^{-\delta t} dt = u\left(\hat{c}_{t_0}\right) + p_{t_0}\left[f\left(k_{t_0}, k_{t_0}\right) - c\left(p_{t_0}\right)\right]$$

Assumption 1 implies

$$\frac{W_{lf} - W_{co}}{p_{t_0}} > \left(\frac{q_{t_0}}{p_{t_0}} - 1\right)\left[f\left(k_{t_0}, k_{t_0}\right) - c\left(p_{t_0}\right)\right] > 0$$

■

In the next section, we show that optimally corrected competitive paths experience a Hopf bifurcation into periodic solutions.

4. Hopf bifurcation

As it is well known, the stationary solution to the command optimum (1) cannot bifurcate into non-constant periodic solutions. The jacobian matrix associated to the command optimum is

$$J \triangleq \begin{bmatrix} \hat{H}_{pk} & \hat{H}_{pp} \\ -\hat{H}_{kk} & -\hat{H}_{pk} + \delta \end{bmatrix}$$

and cannot have a pair of purely imaginary eigenvalues. This follows from several reasons. Take, for instance, the simplest case of constant returns to scale in production analyzed by Benhabib and Nishimura [1] where $\hat{H}_{kk} = 0$. Then J is triangular and its eigenvalues are those of either $\hat{H}_{pk}$ or $\delta - \hat{H}_{pk}$. Both are real, so J cannot have a pair of imaginary eigenvalues.

In the non-constant returns to scale case as well, the existence of a pair of imaginary eigenvalues requires both

$$\hat{H}_{pk}\left(\delta - \hat{H}_{pk}\right) > \hat{H}_{pp}\left(-\hat{H}_{kk}\right)$$

and $tr\,(J) = 0$. This last implies $\delta = 0$, and the above inequality turns out to be

$$-\hat{H}_{pk}^2 > \hat{H}_{pp}\left(-\hat{H}_{kk}\right) \tag{6}$$

This is impossible as long as the direct utility function is concave and the Hamiltonian is strongly concave in k. Although in this case, as Cass and Shell [5] argue, a strictly convex production technology is sufficient to rule out orbiting, the argument put forth by Skiba [9] to prove the impossibility of a limit cycle applies even to the case of a strongly convex (w.r.t. k) Hamiltonian. This follows from continuity of the production, marginal product and consumption functions at a positive quadrant of (k, p). In other words, the above continuity assumptions and strict concavity of the utility function seem to rule out the existence of closed orbits in the aggregative growth model.

In the sequel, we prove that when optimal solutions to the aggregative growth model result from efficient policy aimed at correcting technological externalities, stationary solutions may bifurcate into non-constant periodic paths, continuity and strict concavity assumptions notwithstanding. In order to do so, we investigate the local stability properties of the modified Hamiltonian dynamical system (2) which are described by the matrix:

$$B \triangleq \begin{bmatrix} H_{qk} + H_{qK} & H_{qq} \\ -H_{kk} - H_{kK} & -H_{qk} + \delta - q\frac{\partial \tau}{\partial q} \end{bmatrix} \tag{7}$$

Proof. >From the Hamiltonian

$$H(q,k) = u\left(k, \overset{\circ}{k}\right) + q\left(f(k,K) - c(q) + T\right)$$

We can derive

$$H_q = f(k,K) - c(q) + T$$

$$\overset{\circ}{k}_k = H_{qk} + H_{qK}$$

$$H_{qk} = f_k(k,K) + \tau, \quad H_{qK} = f_K(k,K)$$

$$H_k = u_k + q\left(f_k(k,K) + \tau\right)$$

$$H_{kq} = f_k(k,K) + \tau + q\tau_q = H_{qk} + q\tau_q$$

$$\overset{\circ}{q}_q = \delta - H_{kq}$$

$$\overset{\circ}{q}_k = -H_{kk} - H_{kK}$$

$$H_{kk} = u_{kk} + qf_{kk}(k,K) + q\tau_k, \quad H_{kK} = qf_{kK}(k,K)$$

$$\overset{\circ}{k}_q = H_{qq} = -\frac{\partial c(q)}{\partial q} + T_q$$

■

Some comments are in order. The term H_{qK} is the effect of social (or aggregate) capital on private investment H_q, as in Benhabib and Rustichini [3]. It affects the underlying motion of the real part $\dot{k} = H_q$ of the modified Hamiltonian dynamical system (2). Without corrective policy, the underlying motion would be

$$\dot{k}_k = f_k(k,K) + f_K(k,K)$$

In this case, private productivity is a force increasing the private capital stock, while the negative externality reduces it. With corrective policy, the underlying motion is

$$\dot{k}_k = \frac{p}{q} f_k(k,k) + f_K(k,K)$$

and the force increasing the capital stock becomes the market measure of social marginal productivity.

The term $-q\frac{\partial \tau}{\partial q}$ measures the effect of corrective policy on the underlying motion of the dual part $\dot{q} = \delta q - H_k$ of (2). But from (3) we have

$$-q\tau_q = \frac{p}{q} f_k(k,k)$$

Hence, optimal policy corrects the underlying dynamics of the market price q by the induced (through the tax τ) market measure of social marginal productivity $\frac{p}{q}f_k(k,k)$. This is the way policy ensures social productivity is internalized, i.e. revealed by corrected market.

Since

$$-H_{qk} = -f_k(k,K) - \frac{p}{q}f_k(k,k) + f_k(k,K)$$

it holds that

$$-H_{qk} - q\tau_q = 0$$

We, then, obtain the simpler expression for matrix (7)

$$B \triangleq \begin{bmatrix} H_{qk} + H_{qK} & H_{qq} \\ -H_{kk} - H_{kK} & \delta \end{bmatrix} \tag{8}$$

It turns out that the local dynamical effects of corrective policy are sufficient for the emergence of nontrivial periodic solutions. The following theorem gives a result on the existence of closed orbits under minimal assumptions. In particular, the direct utility function is strictly concave and the Hamiltonian is strongly concave in the private capital stock (the privates' technology is convex).

Theorem 3. *Assume that in the vicinity of the optimal steady state the jacobian matrix of system* (2) *is* (8), *and the following holds*

$$H_{qq}(-H_{kk} - H_{kK}) < -(H_{qk} + H_{qK})^2 \tag{9}$$

Then periodic solutions emerge as the eigenvalues of (8) *cross the real axis.*

Proof. The matrix

$$B = \begin{bmatrix} H_{qk} + H_{qK} & H_{qq} \\ -H_{kk} - H_{kK} & \delta \end{bmatrix} \tag{10}$$

has eigenvalues which are solutions to the polynomial equation

$$\varsigma^2 - (a+d)\varsigma + ad - bc = 0 \tag{11}$$

where

$$a \triangleq H_{qk} + H_{qK} \tag{12}$$
$$d \triangleq \delta \tag{13}$$
$$a + d = H_{qK} + \delta + H_{qk} \tag{14}$$
$$b \triangleq H_{qq} \tag{15}$$
$$c \triangleq -H_{kk} - H_{kK} \tag{16}$$

So we may write

$$\varsigma_{1,2} \triangleq \xi(\delta) \pm i\eta(\delta) \tag{17}$$

where

$$\xi(\delta) \triangleq a + d = \delta + H_{qk} + H_{qK} \tag{18}$$

At the Hopf point, $\delta = \delta^*$ such that

$$\xi(\delta^*) = 0, \tag{19}$$

This implies

$$d = -a \Leftrightarrow \delta = -(H_{qk} + H_{qK}) \tag{20}$$

Then (11) reduces to

$$\varsigma^2 + ad - bc = 0 \tag{21}$$

and

$$\varsigma_{1,2} = \pm i\eta(\delta) \triangleq \pm(bc - ad)^{1/2} \tag{22}$$

where

$$bc - ad = H_{qq}(-H_{kk} - H_{kK}) + (H_{qk} + H_{qK})^2 \tag{23}$$

For the eigenvalues to be purely imaginary, it is needed

$$H_{qq}(-H_{kk} - H_{kK}) < -(H_{qk} + H_{qK})^2 \tag{24}$$

Moreover

$$\left[\frac{\partial \xi(\delta)}{\partial \delta}\right]_{\delta=\delta^*} \neq 0 \tag{25}$$

This completes the proof of bifurcation from a steady solution into periodic orbits (see Iooss and Joseph [6]). ■

It is of interest to discuss what restrictions are imposed on both technology and preferences by condition (24). The fulfillment of (24) is consistent with convex-(joint) concavity of the Hamiltonian as long as either

$$H_{qq} < 0 \; and \; (-H_{kk} - H_{kK}) > 0 \tag{26}$$

or

$$H_{qq} > 0 \; and \; (-H_{kk} - H_{kK}) < 0 \tag{27}$$

In the first case, the policy perturbation is required to make the Hamiltonian strictly concave in the price variable, while preserving (joint) concavity w.r.t. private and social capital. Conversely, the second case refers to a convex-(jointly)convex

Hamiltonian, i.e. $H_{kK} > 0$ and $|H_{kK}| > |H_{kk}|$. This is not equivalent to assuming increasing returns in aggregate capital, but amounts to reducing the degree of concavity in the state variables, as in the example on indeterminacy by Benhabib and Rustichini [3]. Moreover, this is consistent with the case studied by Benhabib and Nishimura [1]. The extent to which the Hamiltonian has to be convex-convex in order to ensure the existence of periodic solutions is determined by the influence of joint capitals on the underlying motion of the real part of the MHDS, $H_{qk} + H_{qK}$. The degree of convex-convexity can be arbitrarily close to zero when the influences of private and aggregate capital on the real underlying motion cancel out each other.

It is worth emphasizing that relaxation of either strong convexity w.r.t. the dual variable, or strong (joint) concavity w.r.t. capital is needed in order to ensure existence of a Hopf point. Yet, in the presence of corrective policy, this requirement is consistent with strict concavity of the direct utility function and strong concavity of the Hamiltonian w.r.t. private capital.

The loss of stability occurs at δ^*, where the eigenvalues of the matrix B become purely imaginary and bifurcation into periodic solutions follows.

In both cases, (18) implies Hopf bifurcation obtains for strictly positive impatience when $H_{qk} + H_{qK} < 0$, i.e. social capital has a negative effect on privates' investment equation which is stronger than the positive effect of private capital.

In particular, we may state

Corollary 4. *Assume $\delta^* > 0$. Then (24) implies $H_{qk} + H_{qK} < 0$.*

Proof. (19) requires

$$\delta^* = -H_{qk} - H_{qK} > 0 \Rightarrow H_{qk} + H_{qK} < 0 \tag{28}$$

■

Remark 5. *The above may seem at odds with the established results in the literature (see Kurz [7]) on the characterization of the eigenvalues associated to the jacobian matrix of competitive paths (as solutions to the inverse optimal growth problem). In particular, Kurz [7] shows that $\delta > 0$ rules out the possibility of purely imaginary eigenvalues. The reason for this is that δ is the trace of matrix J. Matrix B is associated to competitive paths resulting from first-best corrective taxation, which differ from solutions to the inverse optimal problem as long as $p \neq q$. Under this respect, matrix B generalizes matrix J, and $tr\,(B) = \delta + H_{qk} + H_{qK}$ generalizes $tr\,(J) = \delta$. Since for $p = q$ matrix B reduces to J, our results complement those of Kurz [7].*

Remark 6. *Bendixson's criterion is not satisfied since*

$$\delta = -H_{qk} - H_{qK} \Rightarrow \dot{k}_k + \dot{q}_q = 0 \tag{29}$$

In other words, the local divergence of the vector field is identically zero. The occurrence of closed orbit is determined by a nil expansion of flows.

If we relax the assumption of convex technology, we will obtain a version of Skiba [9] model with corrected technological externalities. In that case, the Hamiltonian is convex-convex and (9) holds. As a consequence, the limit cycle in the vicinity of the steady state exists. Skiba [9] applied Bendixson's criterion to show that necessary conditions for the existence of the limit cycle in the optimal model without externalities are not met. Since theorem 1 proves sufficiency for the existence of the limit cycle in the optimal(ly) corrected model with externalities, it might be of interest to complement the analysis by showing how this can be reconciled with Skiba's proof of non-existence (see Skiba [9], p. 533). We do this by proving the following

Corollary 7. *Under conditions of theorem 1, the existence of a limit cycle in the Skiba [9] model is not ruled out by Bendixson's criterion.*

Proof. Upon substitution of $u\left(k, \overset{\circ}{k}\right) = u(c)$, we obtain from (2)

$$\frac{dq}{dk} = \frac{q\left[\delta - f_k(k, K)\right]}{f(k, K) - c(q) + T} \tag{30}$$

or

$$qdf + dqf - dcq + dqT = \delta qdk \tag{31}$$

Suppose the limit cycle exists and integrate the lhs and the rhs along it

$$\oint d(fq) - \oint c(q)\,dq + \oint Tdq = \delta \oint qdk \tag{32}$$

where the rhs is the area of the limit cycle. As in [9], in a market equilibrium $f(k, K)$, $f_k(k, K)$, $c(q)$ are continuous functions at a positive quadrant of (k, q), but, contrasted to [9], T need not be. In fact, it is clear from (3) that we may have the same T for different q, as long as $\frac{p}{q}$ and k are the same. So we obtain that $\oint Tdq$ is not in general zero, and the existence of the limit cycle is not contradicted. ■

We may infer the following intuition form the above results: in the standard optimal aggregate growth model, the dynamics are the result of a command optimum where both the influence of aggregate capital on private investment and of optimal policy on price dynamics disappear. This is since the social planner treats K as a choice variable. As a consequence, a strictly positive effect of impatience on the local expansion of flows cannot be offset. Then, under a command optimum straightforward application of Bendixson's criterion is sufficient to rule out the existence of closed orbits. Nonetheless, when optimal policy corrects the market value of private marginal productivity, the above argument fails.

Periodic solutions may arise for arbitrarily low levels of discounting as well, thus confirming in the one-capital good setting the result obtained by Benhabib and Rustichini [2] for three-sector models. In particular, we have

Corollary 8. *If the system* (2) *is of Liénard type, i.e. the private capital and the social capital effects on the accumulation of private capital offset each other,* $H_{qk} + H_{qK} = 0$*, then periodic solutions arise with zero impatience.*

Proof. It immediately follows from the fact that $\xi(\delta) = \delta$. ■

The reason for orbiting behavior can be explained as follows. In the standard aggregative model, the existence of a single price for investment rules out closed orbits. On the other hand, relative price movements are responsible for the emergence of closed orbits in multisector models. In our optimal path, aggregate investment depends on the relative price $\frac{p}{q}$. Hence, optimal policy introduces a dependence of investment on a relative price by equating private and social rental rates for capital. In other words, the dynamics of the relative price $\frac{p}{q}$ determines an efficient τ modifying the consumption-investment choice in such a way as to generate periodic solutions in the planar system of optimal state and costate variables (k, q).

5. Conclusions

This paper presents an aggregative growth model with technological externalities. First-best policy is used to ensure equality between social and private rental rates for capital. Externalities are corrected by means of non-linear taxes which differ from corrective pigouvian instruments. Since the associated Pareto-optimal paths are distinct from both command optima and solutions to the inverse optimal problem, these possess peculiar local stability properties. The main result is the occurrence of closed orbits as efficient dynamics of the aggregative growth model. The Hopf theorem is applied to prove that the modified Hamiltonian dynamical system experiences a bifurcation from a steady solution into periodic solutions, the standard hypotheses of strictly concave utility and convex privates' technology notwithstanding. Thus, the existence of non-constant periodic solutions as efficient competitive paths is independent of the number of capital goods.

References

[1] Benhabib, J., Nishimura, K., 1979, The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimal Economic Growth, Journal of Economic Theory, 21, 421-444

[2] Benhabib, J., Rustichini, A., 1990, Equilibrium Cycling with Small Discounting, Journal of Economic Theory, 52, 423-432

[3] Benhabib, J., Rustichini, A., 1994, Introduction to the Symposium on Growth, Fluctuations, and Sunspots: Confronting the Data, Journal of Economic Theory, 63, 1-18

[4] Cartigny, P., Venditti, A., 1994, Turnpike Theory: Some New Results on the Saddle Point Property of Equilibria and on the Existence of Endogenous Cycles, Journal of Economic Dynamics and Control, 18, 957-974

[5] Cass, D., Shell, K., 1976, The Structure and Stability of Competitive Dynamical Systems, Journal of Economic Theory, 12, 31-70

[6] Iooss, G., Joseph, D.D., 1981, Elementary Stability and Bifurcation Theory, Springer-Verlag: New York, Heidelberg, Berlin.

[7] Kurz, M., 1968, The General Instability of a Class of Competitive Growth Processes, Review of Economic Studies, 35, 155-174

[8] Medio, A., 1987, Oscillations in Optimal Growth Models, Journal of Economic Behavior and Organization, 8, 413-427

[9] Skiba, A. K., 1978, Optimal Growth with a Convex-Concave production Function, Econometrica, Vol. 46, No. 3, 527-539